4/20 Ⅱ.

1. 定義域，値域を求めよ.

$$z = \log xy$$

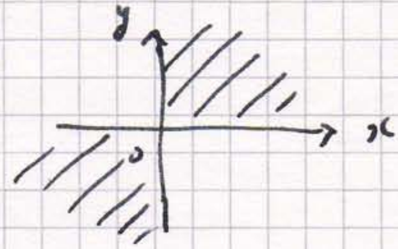
2. 極限値を求めよ.

(1) $$\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$$

(2) $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}.$$

微積II 小テスト　　4/20　　解答.

1.　定義域「$x>0$ かつ $y>0$」または「$x<0$ かつ $y<0$」



値域　実数全体.

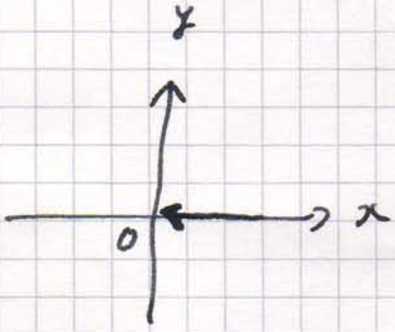
2.(1) 極座標 $x = r\cos\theta$, $y = r\sin\theta$ を代入,

$$\frac{y^3}{x^2+y^2} = r\sin^3\theta \to 0 \quad (r \to 0)$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$$

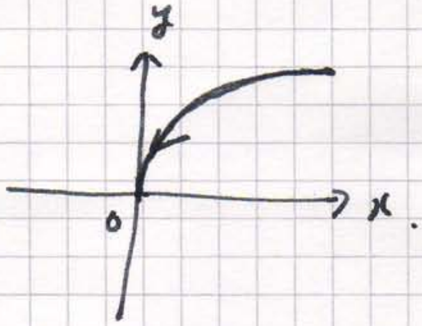
(2)　x軸にそって, $y=0$ とすると

$$\frac{xy^2}{x^2+y^4} = 0$$



$y=\sqrt{x}$ とおくと

$$\frac{xy^2}{x^2+y^4} = \frac{1}{2}$$



よって 極限値なし.

4/27　　　Ⅱ.

1. 偏導関数 $f_x(x,y)$, $f_y(x,y)$ を求めよ.

(1) $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$, $(x,y) \neq (0,0)$

(2) $f(x,y) = e^{-\frac{y}{x}}$, $x \neq 0$

2. 偏微分係数 $f_x(0,0)$ を定義に従って求めよ.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

4/27　　解答.　　　II.

1. (1) $f_x = \dfrac{(x^2y)_x(x^2+y^2) - x^2y(x^2+y^2)_x}{(x^2+y^2)^2}$

$= \dfrac{2xy(x^2+y^2) - x^2y\cdot 2x}{(x^2+y^2)^2}$

$f_y = \dfrac{(x^2y)_y(x^2+y^2) - x^2y(x^2+y^2)_y}{(x^2+y^2)^2} = \dfrac{x^2(x^2+y^2) - x^2y\cdot 2y}{(x^2+y^2)^2}$

(2) $f_x = e^{-\frac{y}{x}} \cdot \dfrac{y}{x^2}$

$f_y = e^{-\frac{y}{x}} \cdot \left(-\dfrac{1}{x}\right)$

2. $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{0-0}{h} \quad \left(\because f(h,0) = \dfrac{h\cdot 0}{h^2+0^2} = 0\right)$

$= 0$

5/11　　II

1. $f(x, y) = \log(e^{x} + e^{-2y})$

について、f_{xy}, f_{yx} を計算して、

$$f_{xy} = f_{yx}$$

を確かめよ。

2. $f(x, y) = x^2 - y^2$ で与えられる曲面

の点 $(1, 1, 0)$ における接平面の方程式

を求めよ。

5/11 解答.

1. $f_{xy} = \dfrac{2e^{x-2y}}{(e^{x}+e^{-2y})^2} = f_{yx}$

2. $f_x = 2x$, $f_y = -2y$

$\therefore z = f_x(1,1)(x-1) + f_y(1,1)(y-1) + f(1,1)$

$= 2x - 2y$

5/18　　II.

1. $z = f(x, y)$ に

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

を合成して、$z = f(r\cos\theta, r\sin\theta)$.

z_θ と $z_{r\theta}$ を求めよ.

2. $z = f(x, y)$, $x = \varphi(t)$, $y = \psi(t)$

のとき 合成関数 $z = f(\varphi(t), \psi(t))$ について、

$\dfrac{dz}{dt}$ を求めよ.

5/18 解答.

1. $z_\theta = f_x x_\theta + f_y y_\theta = f_x \cdot (-r\sin\theta) + f_y \cdot (r\cos\theta)$

$$\boxed{= r(-f_x \sin\theta + f_y \cos\theta)}$$

$$z_r = f_x \cos\theta + f_y \sin\theta$$

$$\begin{aligned}
z_{r\theta} &= (z_r)_\theta \\
&= (f_{xx} \cdot x_\theta + f_{xy} y_\theta)\cos\theta + f_x(-\sin\theta) \\
&\quad + (f_{yx} x_\theta + f_{yy} y_\theta)\sin\theta + f_y \cdot \cos\theta
\end{aligned}$$

(∵ 積の微分公式)

$$\begin{aligned}
&= \left(f_{xx}(-r\sin\theta) + f_{xy} r\cos\theta\right)\cos\theta - f_x \sin\theta \\
&\quad + \left(f_{yx}(-r\sin\theta) + f_{yy} r\cos\theta\right)\sin\theta + f_y\cos\theta \\
&= r\left(-f_{xx}\sin\theta\cos\theta + f_{xy}\cos^2\theta - f_{yx}\sin^2\theta + f_{yy}\sin\theta\cos\theta\right) \\
&\quad - f_x\sin\theta + f_y\cos\theta .
\end{aligned}$$

2. $\dfrac{dz}{dt} = f_x \cdot \varphi'(t) + f_y \cdot \psi'(t)$

5/25　　　Ⅱ.

1.　$f(x, y) = (x^2+1) \log y$

の $(1, e)$ おける $(3, 4)$ 方向の 方向微分係数 を求めよ.

2.　$f(x, y) = x^3 \cos y$　において,

$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^3 f\left(-1, \frac{\pi}{2}\right)$　を求めよ.

5/25 解答.

1. $f_x = 2x \cdot \log y$

$$f_y = (x^2+1)\frac{1}{y} = \frac{x^2+1}{y}$$

$(3,4)$ を $\left(\frac{3}{5}, \frac{4}{5}\right)$ に正規化.

レソーヒガう

$$\frac{3}{5} f_x(1, e) + \frac{4}{5} f_y(1, e) = \frac{3}{5} 2 + \frac{4}{5} \cdot \frac{2}{e}$$

$$= \frac{6}{5} + \frac{8}{5e}.$$

2. $f_x = 3x^2 \cos y$, $f_y = -x^3 \sin y$

$f_{xy} = -3x^2 \sin y$, $f_{xx} = 6x \cos y$, $f_{yy} = -x^3 \cos y$

$f_{xxx} = 6 \cos y$, $f_{xxy} = -6x \sin y$, $f_{xyy} = -3x^2 \cos y$

$f_{yyy} = x^3 \sin y$

$$\therefore \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f\left(-1, \frac{\pi}{2}\right)$$

$$= h^3 \frac{\partial^3 f}{\partial x^3}\left(-1, \frac{\pi}{2}\right) + 3h^2 k \frac{\partial^3 f}{\partial x^2 \partial y}\left(-1, \frac{\pi}{2}\right) + 3hk^2 \frac{\partial^3 f}{\partial x \partial y^2}\left(-1, \frac{\pi}{2}\right)$$

$$+ k^3 \frac{\partial^3 f}{\partial y^3}\left(-1, \frac{\pi}{2}\right)$$

$$= 18h^2 k - k^3.$$

6/8 Ⅱ

1. $f(x,y) = e^{-x} \log(1+y)$

の3次のマクローリンの定理を求めよ.

6/8　解答

$$
\begin{aligned}
f(x, y) = \; & y - xy - \frac{y^2}{2} \\
& - \frac{x^3}{6} e^{-\theta x} \log(1+\theta y) \\
& + \frac{x^2 y}{2} e^{-\theta x} \frac{1}{(1+\theta y)} \\
& + \frac{x y^2}{2} e^{-\theta x} \frac{1}{(1+\theta y)^2} \\
& + \frac{y^3}{3} e^{-\theta x} \frac{1}{(1+\theta y)^3}, \quad 0 < \theta < 1.
\end{aligned}
$$

6/15 II

1. $f(x,y) = x^2 - 3xy + y^3$

の停留点をすべて求めよ.

6/15　解答

$f(x,y) = x^2 - 3xy + y^3$ の停留点をすべて求めよ。

$\boxed{f_x(a,b) = f_y(a,b) = 0}$

$f_x(a,b) = 2x - 3y$

$f_y(a,b) = -3x + 3y^2$

$\boxed{2x - 3y = -3x + 3y^2 = 0}$

$\begin{cases} 2x - 3y = 0 \\ -3x + 3y^2 = 0 \end{cases}$ になる点

$-\cancel{3}x = -\cancel{3}y^2$

$\begin{cases} x = y^2 \\ 2x = 3y \end{cases}$ 代入

$2y^{\cancel{2}} = 3\cancel{y}$

$2y = 3$

$\boxed{y = \frac{3}{2}}$, $\boxed{x = \frac{9}{4}}$

$\therefore \left(\frac{9}{4}, \frac{3}{2}\right)$ $(0, 0)$

6/22　　　II

1. $f(x,y) = x^4 - 4xy + y^2$

の極値をすべて求めよ.

1. $f(x,y) = x^4 - 4xy + y^2$ の極値すべて求めよ.

$f_x = 4x^3 - 4y$

$f_y = -4x + 2y$

$f_x = 0$ かつ $f_y = 0$ より

$4x^3 - 4y = 0 \rightarrow x^3 - y = 0$ — ①

$-4x + 2y = 0 \rightarrow -2x + y = 0$ — ②

①+②: $x^3 - 2x = 0$

$x(x^2 - 2) = 0$

$x(x+\sqrt{2})(x-\sqrt{2}) = 0$

$\underline{x = 0, \pm\sqrt{2}}$

①; $y = x^3$

$\underline{y = 0, \pm 2\sqrt{2}}$

∴ 停留点は $\underline{(x,y) = (0,0), (\sqrt{2}, 2\sqrt{2}), (-\sqrt{2}, -2\sqrt{2})}$

$f_{xx} = 12x^2$

$f_{xy} = -4$

$f_{yy} = 2$

i) $(x,y) = (0,0)$ のとき

$D = 16 - 0 = 16 > 0$ よって極値をもたない

ii) $(x,y) = (\sqrt{2}, 2\sqrt{2})$ のとき

$D = 16 - 24 \cdot 2 < 0$, $f_{xx} = 24 > 0$

よって極小値もつ.

$f(\sqrt{2}, 2\sqrt{2}) = 4 - 4 \cdot 4 + 8 = -4$

$\underline{∴ (x,y) = (\sqrt{2}, 2\sqrt{2})\ のとき\ 極小値\ -4}$

iii) $(x,y) = (-\sqrt{2}, -2\sqrt{2})$ のとき

$D = 16 - 24 \cdot 2 < 0$, $f_{xx} = 24 > 0$ よって極小値もつ.

$f(-\sqrt{2}, -2\sqrt{2}) = 4 - 4 \cdot 4 + 8 = -4$

$\underline{∴ (x,y) = (-\sqrt{2}, -2\sqrt{2})\ のとき極小値\ -4}$

6/29　　II

1.　重積分の単調性から次の不等式を導け.

$$\left| \iint_D f(x,y)\,dxdy \right| \leqq \iint_D |f(x,y)|\,dxdy$$

2.　$f(x,y) \geqq 0$ とするとき,

$\iint_D f(x,y)\,dxdy = 0$ が成立すれば $f(x,y)=0$ に限ることを示せ.

6/29 解答

1. $-|f(x,y)| \leqq f(x,y) \leqq |f(x,y)|$

より単調性から

$$-\iint_D |f(x,y)|\,dxdy \leqq \iint_D f(x,y)\,dxdy \leqq \iint_D |f(x,y)|\,dxdy$$

$$\therefore \left|\iint_D f(x,y)\,dxdy\right| \leqq \iint_D |f(x,y)|\,dxdy.$$

2. $f(x,y)=0$ ならば $\iint_D f(x,y)\,dxdy = 0$.

逆に

$\iint_D f(x,y)\,dxdy = 0$ のとき, $f(x,y) > 0$ をみたす

(x,y) が存在すれば, 体積の概念より $\iint_D f(x,y)\,dxdy$ は

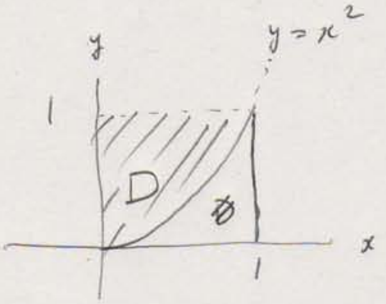
正である. よって $f(x,y)=0$ に限る

7/6　　Ⅱ

1. $\iint_D x^3 y \, dx\, dy$

$D = \{ (x, y) \mid 0 \leqq x \leqq 1 ,\quad 1 \leqq y \leqq 3 \}$

2.



$\iint_D f(x, y) \, dx\, dy$

(1) y, x の順に 累次積分せよ

(2) x, y の順に 〃

1 $\iint_D x^3 y \, dx\, dy$

$D = \{(x,y) \mid 0 \leqq x \leqq 1, 1 \leqq y \leqq 3\}$

を求めよ.

$\int_0^1 \left(\int_1^3 x^3 y \, dy\right) dx$

$\int_0^1 \left(x^3 \left[\frac{1}{2}y^2\right]_1^3\right) dx$

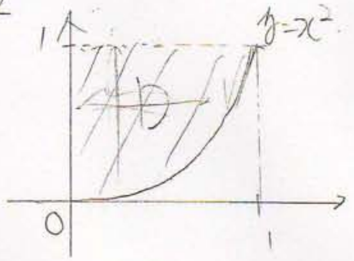
$x^3 \left(\frac{9}{2} - \frac{1}{2}\right) =$

$\int_0^1 4x^3 \, dx$

$\left[x^4\right]_0^1 = 1 - 0$

$= 1$

2



$\iint_D f(x,y)\, dx\, dy$ について

(1) y、xの順に累次積分

$D = \{(x,y) \mid 0 \leqq x \leqq 1, x^2 \leqq y \leqq 1\}$

$\int_0^1 \left(\int_{x^2}^1 f(x,y)\, dy\right) dx$

(2) x、yの順に累次積分

$D = \{(x,y) \mid 0 \leqq y \leqq 1, 0 \leqq x \leqq \sqrt{y}\}$

$\int_0^1 \left(\int_0^{\sqrt{y}} f(x,y)\, dx\right) dy$

7/13 II

1. 積分順序の交換をせよ.

$$\int_0^1 \left(\int_0^x f(x,y)\,dy \right) dx + \int_1^2 \left(\int_0^{2-x} f(x,y)\,dy \right) dx$$

2. $$\int_0^{\pi} \left(\int_0^x y \cos(x-y)\,dy \right) dx$$

(1) 値を求めよ.

(2) 順序を交換して値を求めよ.

1、交換

$$\int_0^1 \left(\int_0^x f(x,y)\,dy \right) dx + \int_1^2 \left(\int_0^{2-x} f(x,y)\,dy \right) dx$$

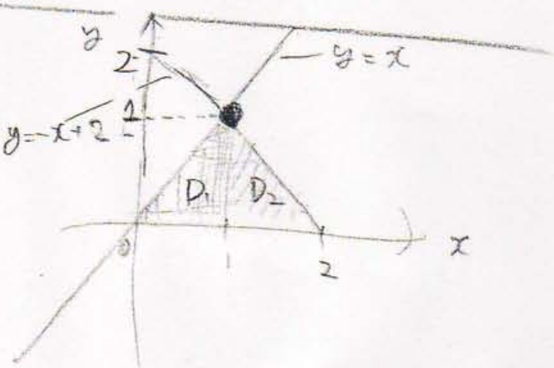
2、$\int_0^{\pi} \left(\int_0^x y\cos(x-y)\,dy \right) dx$

(1) 値

(2) 交換して値

1.

$$与式 = \int_0^1 \left(\int_y^{2-y} f(x,y)\,dx \right) dy$$



y=-x+2

2、

(1) $\int_0^{\pi} \left(\left[-y\sin(x-y) \right]_0^x + \int_0^x \sin(x-y)\,dy \right) dx$

$= \int_0^{\pi} \left(\left[+\cos(x-y) \right]_0^x \right) dx$

$= \int_0^{\pi} (1 - \cos x)\,dx$

$= \left[x - \sin x \right]_0^{\pi}$

$= \pi - 0 - (0 - 0) = \pi$

(2) $\int_0^{\pi} \left(\int_y^{\pi} y\cos(x-y)\,dx \right) dy$

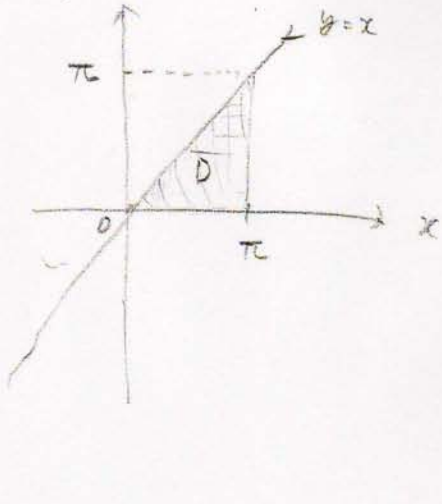
$= \int_0^{\pi} \left[y\sin(x-y) \right]_y^{\pi} dy$

$= \int_0^{\pi} y\sin(\pi-y)\,dy$

$= \left[y\cos(\pi-y) \right]_0^{\pi} - \int_0^{\pi} \cos(\pi-y)\,dy$

$= \pi - \left[-\sin(\pi-y) \right]_0^{\pi}$

$= \pi$



7/20 Ⅱ

1. $$\iint_D (x-y)^2\,dx\,dy,$$

$$D = \{(x,y) \mid -1 \leqq x+2y \leqq 1,\ -1 \leqq x-y \leqq 1\}$$

変数変換を用いて求めよ.

1. $\iint_D (x-y)^2\,dx\,dy$ 　　変数変換を用いて、値を求めよ

$$D = \left\{ (x,y) \;\middle|\; \begin{array}{l} -1 \leq x+2y \leq 1 \\ -1 \leq x-y \leq 1 \end{array} \right\}$$

$$\begin{cases} u = x+2y \\ v = x-y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}u + \frac{2}{3}v \\ y = \frac{1}{3}u - \frac{1}{3}v \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

$$D = \left\{ (u,v) \;\middle|\; \begin{array}{l} -1 \leq u \leq 1 \\ -1 \leq v \leq 1 \end{array} \right\}$$

$$\iint_D \left\{ \left(\frac{1}{3}u + \frac{2}{3}v\right) - \left(\frac{1}{3}u - \frac{1}{3}v\right) \right\}^2 \left|-\frac{1}{3}\right| du\,dv$$

$$= \iint_D \frac{1}{3}v^2\,du\,dv$$

$$\int_{-1}^{1} \left(\int_{-1}^{1} \frac{1}{3}v^2\,dv \right) du$$

$$= \int_{-1}^{1} \left[\frac{1}{9}v^3 \right]_{-1}^{1} du$$

$$= \int_{-1}^{1} \left(\frac{1}{9} + \frac{1}{9} \right) du$$

$$= \int_{-1}^{1} \frac{2}{9}\,du$$

$$= \left[\frac{2}{9}u \right]_{-1}^{1}$$

$$= \frac{4}{9}$$

解答