[1] (1)

$$S(x) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b).$$

(2)

$$f_x(x,y) = \frac{y}{x^2}\sin\frac{y}{x}, \quad f_y(x,y) = -\frac{1}{x}\sin\frac{y}{x}.$$

よって $f_x(2,\pi) = \frac{\pi}{4}$, $f_y(2,\pi) = -\frac{1}{2}$, $f(2,\pi) = 0$ だから

$$S(x) = \frac{\pi}{4}(x-2) - \frac{1}{2}(y-\pi) = \frac{\pi}{4}x - \frac{1}{2}y.$$

[2] (1)

$$\begin{aligned} & f(a+h, b+k) \\ = \; & f(a,b) + f_x(a,b)h + f_y(a,b)k \\ & + \frac{1}{2}(f_{xx}(a+\theta h, b+\theta k)h^2 + 2f_{xy}(a+\theta h, b+\theta k)hk + f_{yy}(a+\theta h, b+\theta k)k^2), \quad 0 < \theta < 1. \end{aligned}$$

(2)

$$\begin{aligned} & f(1,1) = \frac{\log 4}{e}, \\ & f_x(x,y) = e^{-y}\frac{3}{1+3x}, \quad f_x(1,1) = \frac{3}{4e}, \\ & f_y(x,y) = -e^{-y}\log(1+3x), \quad f_y(1,1) = -\frac{\log 4}{e}, \\ & f_{xx}(x,y) = -e^{-y}\frac{9}{(1+3x)^2}, \\ & f_{xy}(x,y) = -e^{-y}\frac{3}{1+3x}, \\ & f_{yy}(x,y) = e^{-y}\log(1+3x). \end{aligned}$$

よって

$$\begin{aligned} & e^{-(1+h)}\log(1+3(1+k)) \\ = \; & \frac{\log 4}{e} + \frac{3}{4e}h - \frac{\log 4}{e}k \\ & + \frac{1}{2}\left(-e^{-(1+\theta k)}\frac{9}{(1+3(1+\theta h))^2})h^2 - 2e^{-(1+\theta k)}\frac{3}{1+3(1+\theta h)}hk \right. \\ & \quad \left. + e^{-(1+\theta k)}\log(1+3(1+\theta h))k^2\right), \quad 0 < \theta < 1. \end{aligned}$$

[3]

$$f_x = 14(2x-y)^6 + 3x^2y^5 \quad f_y = -7(2x-y)^6 + 5x^3y^4.$$

$$x_\theta = -r\sin\theta, \quad y_\theta = r\cos\theta.$$

よって

$$z_\theta = f_x x_\theta + f_y y_\theta = -\{14(2x-y)^6 + 3x^2y^5\}r\sin\theta + \{-7(2x-y)^6 + 5x^3y^4\}r\cos\theta.$$

$r = 1$, $\theta = \pi$ のとき，$x = -1$, $y = 0$. よって

$$z_\theta(1,\pi) = 448.$$

[4] 停留点を求める．

$$f_x = 4x^3 - 4y = 0, \quad f_y = -4x + 2y = 0.$$

第二式の $y = 2x$ を第一式に代入．$4x^3 - 8x = 0$. ゆえに $x = 0, \pm\sqrt{2}$. 停留点は

$$(x, y) = (0, 0),\ (\sqrt{2}, 2\sqrt{2}),\ (-\sqrt{2}, -2\sqrt{2}).$$

続いて

$$f_{xx} = 12x^2, \quad f_{xy} = -4, \quad f_{yy} = 2.$$

$$\therefore\ \Delta = f_{xy}^2 - f_{xx}f_{yy} = 16 - 24x^2.$$

$$\Delta(0, 0) = 16 > 0.$$

よって $f(0, 0)$ は極値ではない．

$$\Delta(\sqrt{2}, 2\sqrt{2}) = -32 < 0, \quad f_{xx}(\sqrt{2}, 2\sqrt{2}) = 24 > 0$$

だから $f(\sqrt{2}, 2\sqrt{2}) = -4$ は極小値.

$$\Delta(-\sqrt{2}, -2\sqrt{2}) = -32 < 0, \quad f_{xx}(-\sqrt{2}, -2\sqrt{2}) = 24 > 0$$

だから $f(-\sqrt{2}, -2\sqrt{2}) = -4$ は極小値.