Blowing-up properties of the positive principal eigenvalue for indefinite Robin-type boundary conditions

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Stability of the equilibrium $u \equiv 0$

- For a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$,

$$
\begin{cases}
\frac{\partial u}{\partial t}(t,x) = d\,\Delta u + g(x)\, u - u^2 & , t > 0, \quad x \in \Omega, \\
u(0,x) = u_0(x) \geq 0 & , x \in \Omega, \\
(d\,\nabla u) \cdot \mathrm{n} = h(x)\, u & , t > 0, \quad x \in \partial\Omega.
\end{cases}
$$

Here: $d > 0$, $g \in L^\infty(\Omega)$, h is smooth,

$g > 0$ on some set of positive measure ($\|g\|_\infty > 0$).

- The linearized eigenvalue problem at the equilibrium $u \equiv 0$ is that

(1)
$$
\begin{cases}
-d\,\Delta\phi = g(x)\phi + \gamma(d)\,\phi & , x \in \Omega, \\
(d\,\nabla\phi) \cdot \mathrm{n} = h(x)\phi & , x \in \partial\Omega.
\end{cases}
$$

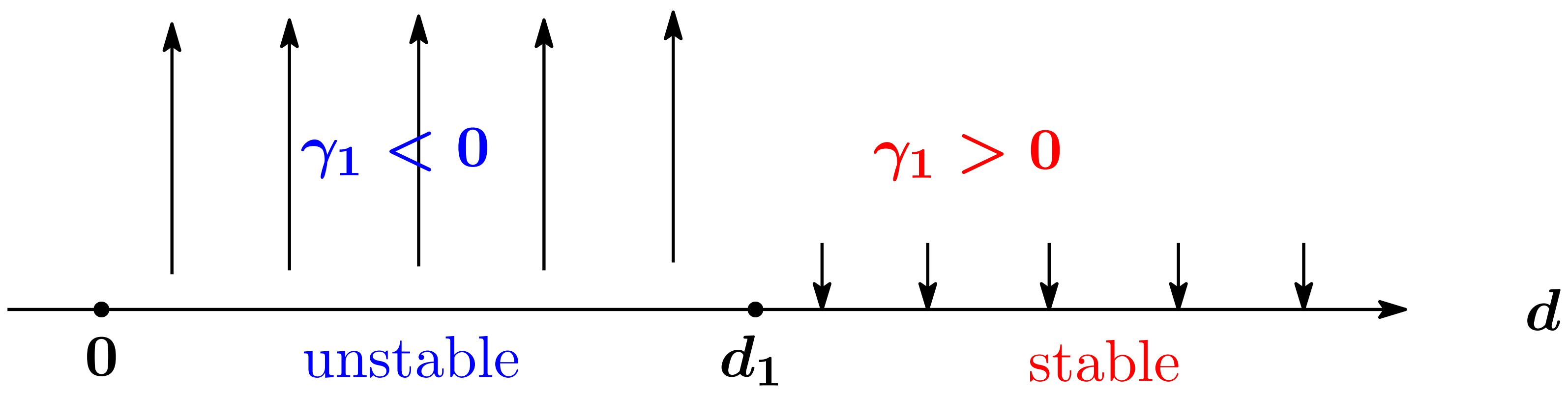
- Putting $\boldsymbol{\lambda = d^{-1}}$ gives

$$(2)\quad \begin{cases} -\Delta\phi = \lambda\, g(x)\phi + \mu(\lambda)\phi & , x \in \Omega, \\ (\nabla\phi)\cdot \mathbf{n} = \lambda\, h(x)\phi & , x \in \partial\Omega. \end{cases}$$

- $\displaystyle \int_\Omega g\, dx + \int_{\partial\Omega} h\, ds < 0 \implies \begin{cases} \mu_1 > 0 & \text{if } 0 < \lambda < \exists\, \lambda_1, \\ \mu_1 < 0 & \text{if } \lambda > \lambda_1. \end{cases}$

- $\mu_1(\lambda)$ for $\lambda > \lambda_1$ $\sim$ $\gamma_1(d)$ for $d < d_1 := \lambda_1^{-1}$

(Note that $\displaystyle \mu_1(\lambda) = \frac{\gamma_1(d)}{d}$)



Blowing-up behavior of λ_1 (OUR INTEREST)

- Which families of g and h determine the blowing-up behavior

$$\lambda_1(g,\ h) \longrightarrow +\infty \quad (\text{that is, } d_1 \longrightarrow +0) \quad ?$$

- Find WORST environment for the species.

A related result due to Cantrell-Cosner (Dirichlet case)

- Under $\sup_j \|g_j\|_\infty < \infty$, Cantrell-Cosner ('89) gave a necessary and sufficient condition for the blowing-up, which is as follows.

(3)

$$\overline{\lim_{j\to\infty}} \int_\Omega g_j(x)\psi \, dx \leq 0, \qquad \forall \psi \in L^1(\Omega) \text{ s.t. } \psi \geq 0 \text{ a.e.}$$

- Note that

$$\| (g_j)^+ \|_\infty \longrightarrow 0 \qquad \Longrightarrow \qquad (3).$$

An extension to the indefinite Robin case (Main result)

THEOREM 1 Let

$$\sup_j \|g_j\|_\infty < \infty, \qquad \sup_j \|h_j\|_\infty < \infty,$$

be satisfied. Then, the condition

(4)

$$\begin{cases} \overline{\lim_{j\to\infty}} \int_\Omega g_j \psi \, dx \leq 0, & \forall \psi \in L^1(\Omega) \text{ s.t. } \psi \geq 0 \text{ a.e.,} \\ \overline{\lim_{j\to\infty}} \int_{\partial\Omega} h_j \phi \, ds \leq 0, & \forall \phi \in L^1(\partial\Omega) \text{ s.t. } \phi \geq 0 \text{ a.e.} \end{cases}$$

is <u>necessary and sufficient for the blowing-up</u>, provided

(5)
$$\overline{\lim_{j\to\infty}} \left(\int_\Omega g_j \, dx + \int_{\partial\Omega} h_j \, ds \right) < 0.$$

Non blowing-up result induced by a counterexample

- What happens in case: $\displaystyle \int_{\Omega} g_j \, dx \; + \; \int_{\partial\Omega} h_j \, ds \;\nearrow\; 0$?

- $h_j \equiv 0 \implies \exists\, g_j \in L^{\infty}(\Omega)$ s.t.

(6) $$\begin{cases} \| (g_j)^+ \|_{\infty} \longrightarrow 0 \,, \\ \lambda_1 (g_j, 0) \quad \text{is bounded above} \,. \end{cases}$$

- Note that

(7) $$\int_{\Omega} g_j \, dx \;\nearrow\; 0 \quad \Longleftrightarrow \quad \theta_j := \frac{\displaystyle\int_{\Omega} (g_j)^- \, dx}{\displaystyle\int_{\Omega} (g_j)^+ \, dx} \;\searrow\; 1.$$

- the speed $\theta_j \longrightarrow 1$, low ? high ?

$\int_I (g_j)^+ \, dx$ VERSUS $\int_I (g_j)^- \, dx$ (Main result)

THEOREM 2 • Let $I \subset \mathbb{R}$ be an open interval, $h_j \equiv 0$.

- ASSUME some class of g_j s.t. $\int_I g_j \, dx \nearrow 0$.
- If we have

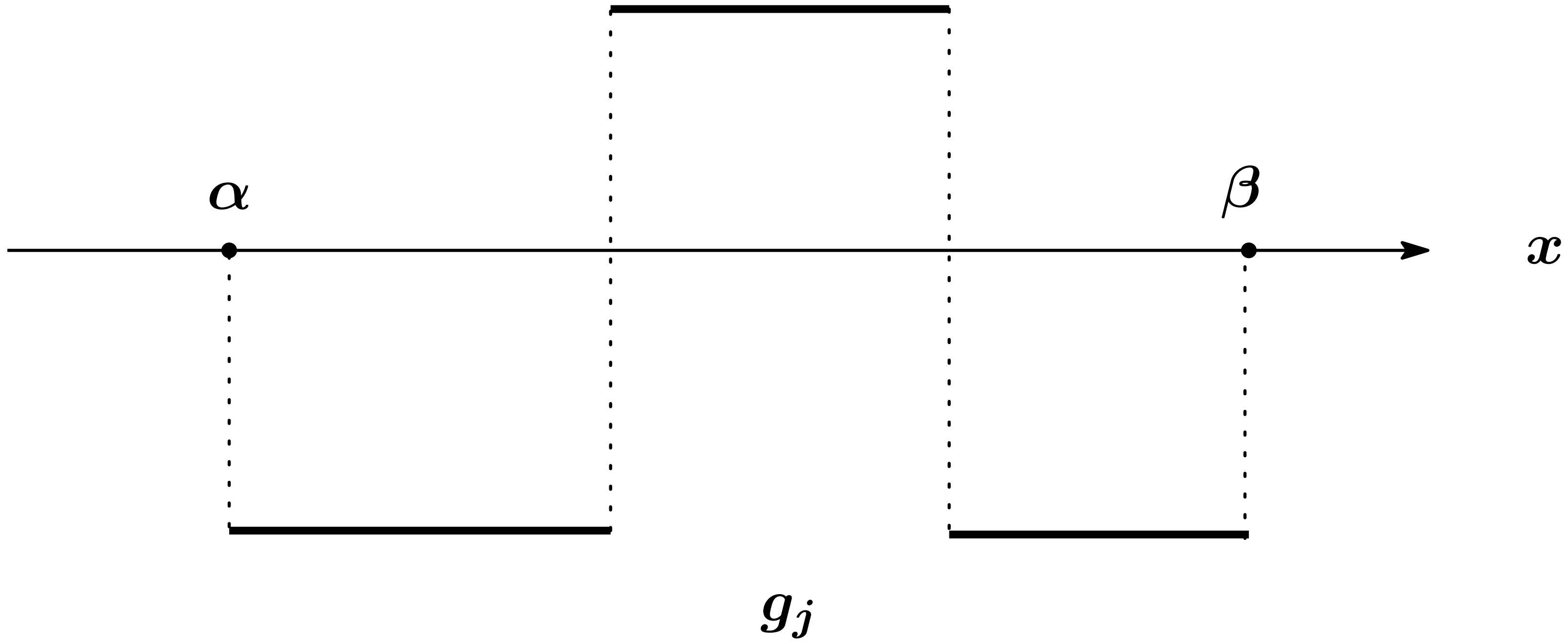
$$\theta_j = 1 + C \left(\int_I (g_j)^+ \, dx \right)^{\sigma} + \underbrace{\cdots\cdots}_{\text{higher order terms}} \tag{8}$$

as $\int_I (g_j)^+ \, dx \searrow 0$, with some $C > 0$,

then

$$\begin{cases} \sigma \geq 1 & \Longrightarrow \quad \lambda_1(g_j, 0) \text{ is bounded above (survival)} \\ 0 < \sigma < 1 & \Longrightarrow \quad \lambda_1(g_j, 0) \text{ blows up (extinct)} \end{cases} \tag{9}$$

Class of g_j supporting the assertion (9)



- step function g_j whose positive part is given by a sub-interval of (α, β)

Thank you for your attention.

Variational characterization of $\boldsymbol{\lambda_1}$

$$\boldsymbol{\lambda_1}\ (g,\ h)\ =\ \inf\left\{ \frac{\displaystyle\int_\Omega |\nabla\varphi|^2 dx}{\displaystyle\int_\Omega g\varphi^2 dx + \int_{\partial\Omega} h\varphi^2 ds} :\ \varphi \in W^{1,2}(\Omega),\right.$$

$$\left.\int_\Omega g\varphi^2 dx + \int_{\partial\Omega} h\varphi^2 ds > 0 \right\}$$