On S-shaped and CS-shaped bifurcation diagrams in population dynamics

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梅 津 健一郎（茨城大教育）
H.R. Quoirin (Univ. de Santiago de Chile)

Problem Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a smooth bounded domain. Consider

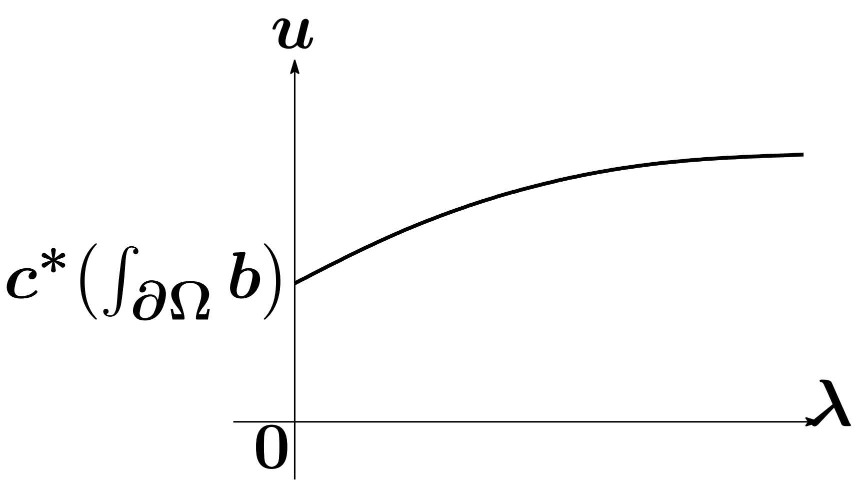
$$\begin{cases} -\Delta u = \lambda(m(x)u - a(x)|u|^{p-2}u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}} = \lambda b(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

- $\lambda \in \mathbb{R}$ is a parameter.
- m, b may change sign.
- $m^+ \not\equiv 0$.
- $a > 0$.

$$1 < q < 2 < p \quad (q - 1 < 1 < p - 1).$$

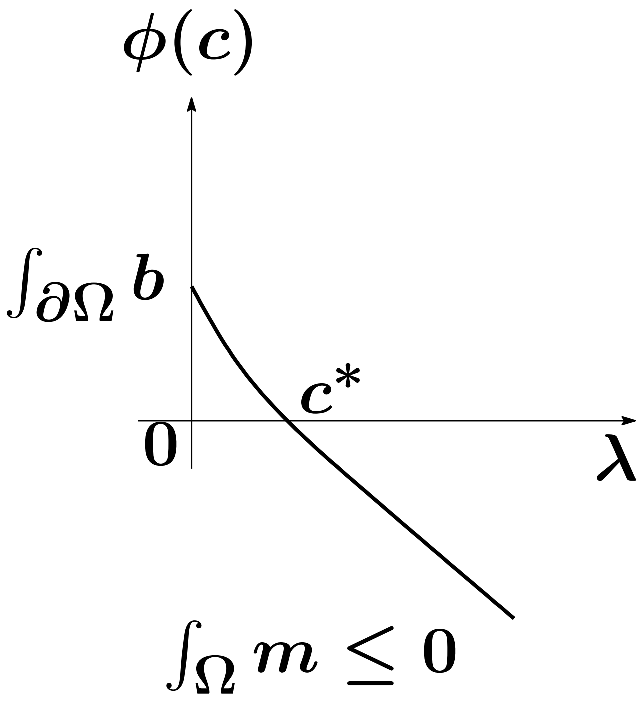
If $b \geq 0$ then positive solution $u > 0$ in $\overline{\Omega}$ is unique for every $\lambda > 0$ (Pao '93).

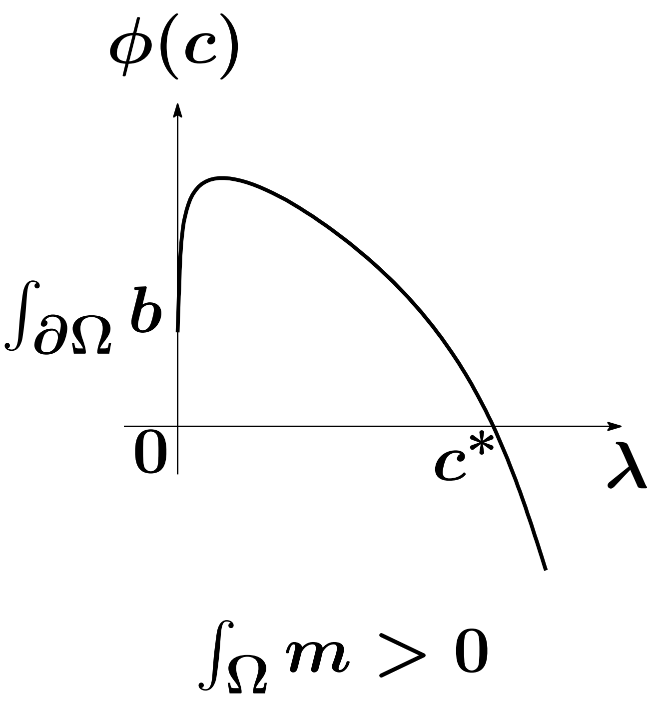
Case $b > 0$ (García-Melìan, Morales-Rodrigo, Rossi, and Suárez '08).



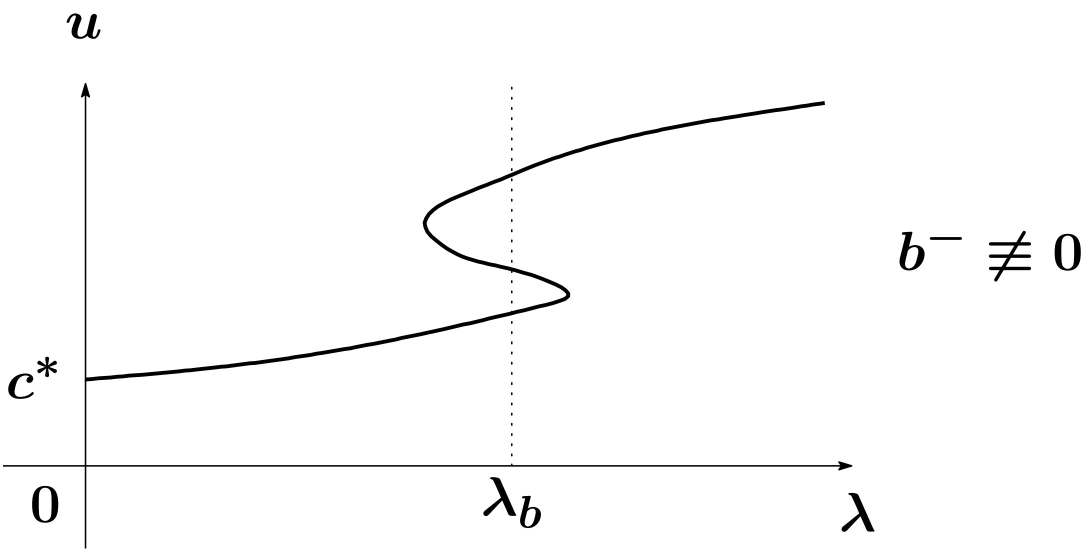
$c^*(\int_{\partial\Omega} b)$ is the positive unique zero of

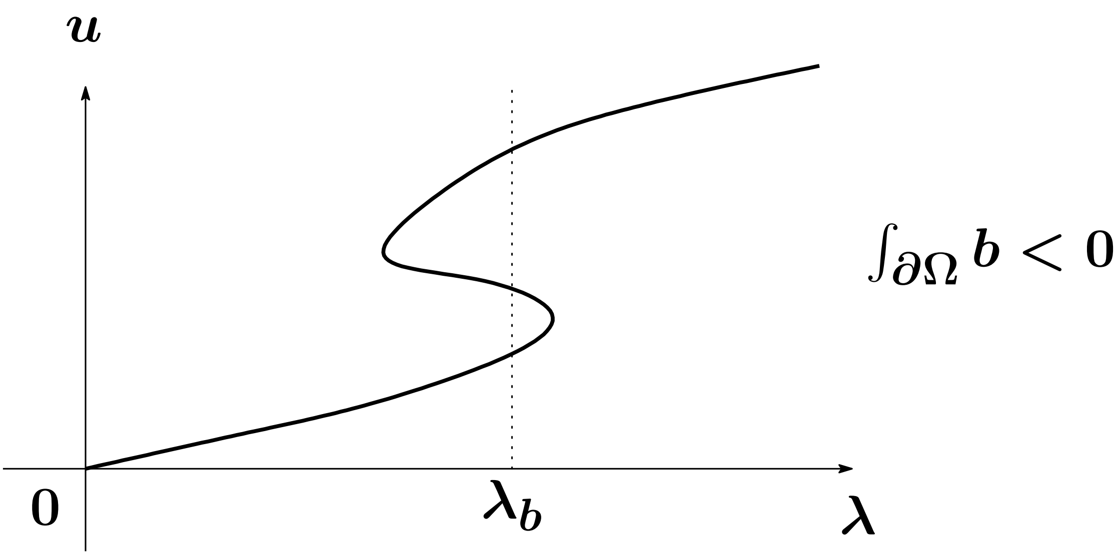
$$\phi(c) := c^{2-q}\int_\Omega m - c^{p-q}\int_\Omega a + \int_{\partial\Omega} b.$$

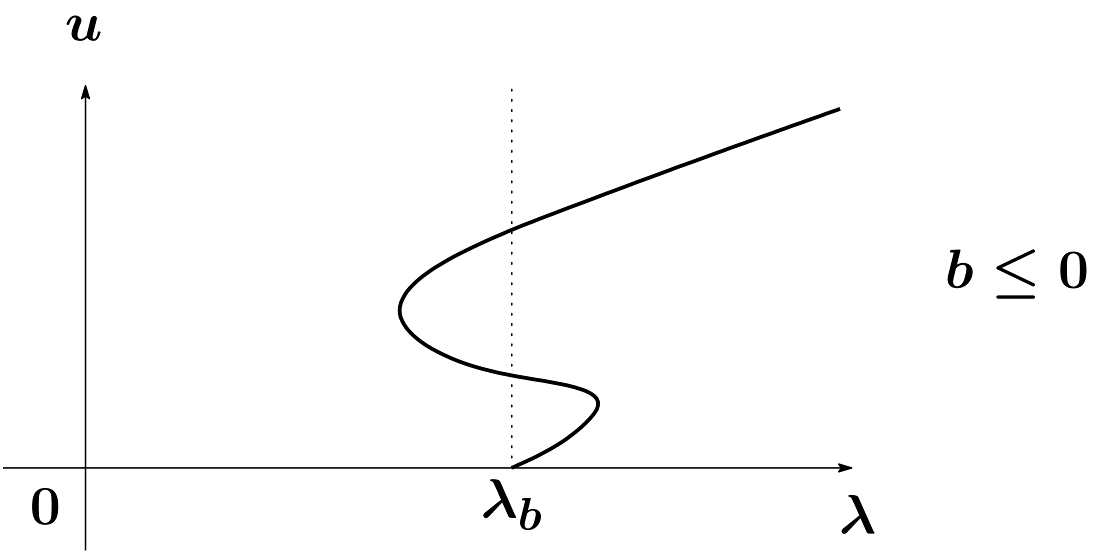




A heuristic observation if the negative part of b increases.







$$\lambda_1(m) = \inf\{\int_\Omega |\nabla u|^2 : u \in H^1(\Omega), \int_\Omega m u^2 = 1\}$$

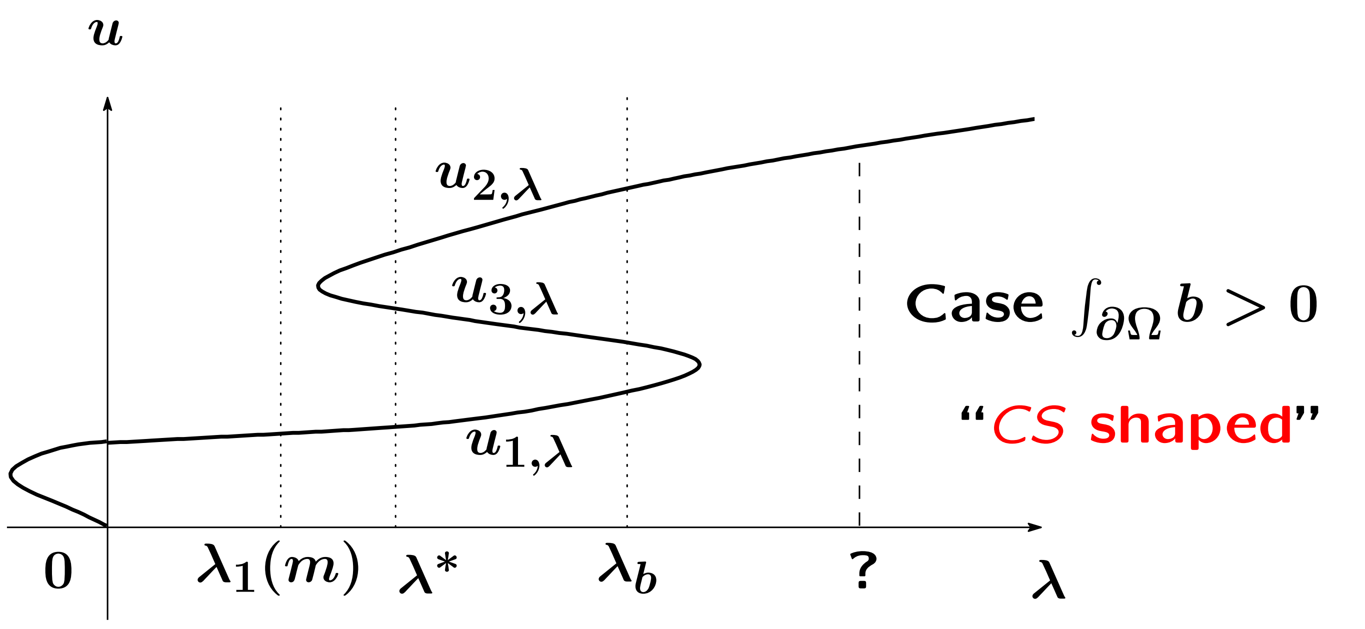
$$\lambda_b = \inf\{\int_\Omega |\nabla u|^2 : u \in H^1(\Omega), \int_\Omega m u^2 = 1, \int_{\partial\Omega} b|u|^q \geq 0\}$$

Remark $\lambda_b = \lambda_1(m)$ if $b \geq 0$, whereas $\lambda_b = \lambda_1^D(m)$ (Dirichlet b.c.) if $b < 0$.

Theorem 1(2) Let $p \leq \frac{2N}{N-2}$ if $N > 2$. Assume $b^+ \not\equiv 0$. Then, the problem has at least 1 nontrivial nonnegative solution for every $\lambda > 0$. Additionally assume

$$\int_\Omega m < 0 \quad \text{and} \quad \int_{\partial\Omega} b\varphi_1^q < 0.$$

Then, $\lambda_1(m) < \lambda_b$, and if $\int_\Omega a\varphi_1^p$ is small then there exists $\lambda^*(a) \in (\lambda_1(m), \lambda_b)$ such that the problem has at least 3 nontrivial nonnegative solutions for $\lambda \in (\lambda^*, \lambda_b)$.



Sketch of Proof For $u \in H^1(\Omega)$, set

$$I_\lambda(u) = \frac{1}{2}E_\lambda(u) + \frac{\lambda}{p}A(u) - \frac{\lambda}{q}B(u),$$

where

$$E_\lambda(u) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega m u^2,$$
$$A(u) = \int_\Omega a|u|^p, \qquad B(u) = \int_{\partial\Omega} b|u|^q.$$

- $\exists u_{1,\lambda} \in B^+$ s.t.

$$I_\lambda(u_{1,\lambda}) = \inf_{u \in B^+} I_\lambda(u) < 0$$

if $\lambda \in (0, \lambda_b)$, where

$$B^+ = \{u \in H^1(\Omega) : B(u) > 0\}.$$

- $\exists u_{2,\lambda} \in E_\lambda^-$ s.t.

$$I_\lambda(u_{2,\lambda}) = \inf_{u \in E_\lambda^-} I_\lambda(u)$$

if $\lambda \in (0, \lambda_b)$ and $\inf_{u \in E_\lambda^-} I_\lambda(u) < 0$, where

$$E_\lambda^- = \{u \in H^1(\Omega) : E_\lambda(u) < 0\}$$

- $\lambda \in (0, \lambda_b) \Longrightarrow B^+ \cap E_\lambda^- = \emptyset$

- mountain pass level $c_\lambda > 0 > I_\lambda(u_{1,\lambda}) \vee I_\lambda(u_{2,\lambda})$
$\Longrightarrow$ the third solution $u_{3,\lambda}$

The verification of

$$\inf_{u\in B^+} I_\lambda(u) < 0 \quad \text{and} \quad \inf_{u\in E_\lambda^-} I_\lambda(u) < 0$$

- Since $b^+ \not\equiv 0$, we have for some u_0, satisfying $B(u_0) > 0$, and $t > 0$ small,

$$I_\lambda(tu_0) = \frac{t^2}{2}E_\lambda(u_0) + \frac{\lambda t^p}{p}A(u_0) - \frac{\lambda t^q}{q}B(u_0) < 0.$$

- Note $E_\lambda(t\varphi_1) = t^2(\lambda_1 - \lambda)\int_\Omega m\varphi_1^2 < 0$ for $\lambda > \lambda_1$. Consider

$$\begin{aligned}\psi(t) &= \frac{I_\lambda(t\varphi_1)}{t^q} \\ &= -\frac{\lambda}{q}B(\varphi_1) + \frac{t^{2-q}}{2}(\lambda_1 - \lambda) + \frac{\lambda t^{p-q}}{p}A(\varphi_1) < 0.\end{aligned}$$

ψ has the global minimum

$$\psi(t_0) = -\frac{1}{q}\left(\lambda B(\varphi_1) + C_{pq}\frac{(\lambda-\lambda_1)^{\frac{p-q}{p-2}}}{(\lambda A(\varphi_1))^{\frac{2-q}{p-2}}}\right).$$

We see

$$\begin{aligned}&\psi(t_0) < 0 \iff \\ &\left(1 - \frac{\lambda_1}{\lambda}\right)^{\frac{p-q}{p-2}} > C_{pq}^{-1}\left(-B(\varphi_1)\right) A(\varphi_1)^{\frac{2-q}{p-2}}.\end{aligned}$$