Exact multiplicity of positive solutions for an indefinite concave Robin bvp

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問題設定

$\Omega \subset \mathbb{R}^N$, $N \geq 1$ は滑らかな境界をもつ有界領域.

$$(P_\alpha) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ \partial_\nu u = \alpha u & \text{on} \quad \partial\Omega. \end{cases}$$

- $a \in C(\overline{\Omega})$ は sign-changing.
- $0 < q < 1 \quad$ (concave)
- ν は $\partial\Omega$ 上外向き単位法線ベクトル場, $\partial_\nu = \frac{\partial}{\partial \nu}$.
- $\alpha \in [-\infty, \infty)$ は parameter. $\alpha = -\infty$ のときは, $u|_{\partial\Omega} = 0$.

$$(P_\alpha) \text{ の解 } u \iff (P_\alpha) \text{ を満たす } u \in W^{2,r}(\Omega),\ r > N.$$

強正値性

- $q \geq 1$ or $a(x) \geq 0$ の場合,

$$\text{非自明解} \quad u \not\equiv 0 \quad \Longrightarrow \quad u \gg 0 \text{ (強正値性) :}$$

$$u \in \begin{cases} \{u \in C^1(\overline{\Omega}) : u > 0 \text{ in } \overline{\Omega}\} & (\alpha \neq -\infty) \\ \{u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega,\ \partial_\nu u < 0 \text{ on } \partial\Omega\} & (\alpha = -\infty) \end{cases}$$

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- $0 < q < 1$ and $a(x)$ changes sign の場合,

$$\text{非自明解} \quad \exists u \not\equiv 0 \quad \text{s.t.} \quad u \not\gg 0.$$

強正値性が壊れる例

Set $\Omega := (0, \pi)$, $0 < q < 1$, $r := \frac{2}{1-q} > 2$, and

$$a(x) := r^{1-\frac{2}{r}} \left(1 - r\cos^2 x\right).$$

Then, $u(x) = \dfrac{\sin^r x}{r}$ solves

$$\begin{cases} -u''(x) = a(x)u(x)^q, & x \in \Omega, \\ u(x) = u'(x) = 0, & x = 0,\ \pi, \end{cases}$$

and satisfies

$$u > 0 \ \text{ in } \Omega, \quad \text{but} \quad u \not\gg 0.$$

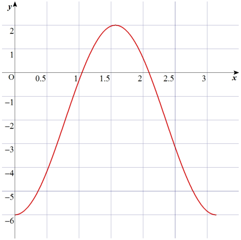

Figure: The weight a with $q = \frac{1}{2}$.

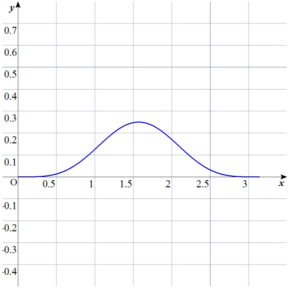

Figure: The solution $u > 0$ in the case $q = \frac{1}{2}$.

強正値性が成り立つ $q \in (0,1)$

$$\mathcal{A}_\alpha(a) := \{q \in (0,1) : \text{非自明解 } u \not\equiv 0 \implies u \gg 0\}.$$

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$$\Omega_+^a := \{x \in \Omega : a(x) > 0\} \text{ has finitely many components.} \tag{A.1}$$

$$\partial\Omega \subset \partial\Omega_+^a. \tag{A.2}$$

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命題 1. (1) (Connectivity) Assume $(A.1)$. Then, $\exists q_{-\infty}, q_0 \in [0,1)$ s.t.

$$\mathcal{A}_{-\infty} = (q_{-\infty}, 1), \quad \mathcal{A}_0 = (q_0, 1) \text{ (provided that } \textstyle\int_\Omega a < 0).$$

(2) (Monotonicity)

$$\mathcal{A}_\alpha \subset \mathcal{A}_\beta \,(-\infty \le \alpha < \beta \le 0); \quad \mathcal{A}_\alpha \subset \mathcal{A}_\beta \,(0 \le \alpha < \beta < \alpha_p),$$

$$\alpha_p := \inf\left\{\int_\Omega |\nabla v|^2 : v = 0 \text{ in } \Omega_+^a,\ \int_{\partial\Omega} v^2 = 1\right\}.$$

注意. $\alpha_p = \infty \iff (A.2)$ holds.

定理 1 ($\alpha \leq 0$)

Assume $(A.1)$ and $q \in \mathcal{A}_{-\infty}$. Then, (P_α) has a *unique* nontrivial solution $u_1(\alpha) \gg 0$ for $\alpha \in [-\infty, 0)$ if $\int_\Omega a \geq 0$ and for $\alpha \in [-\infty, 0]$ if $\int_\Omega a < 0$, satisfying the conditions as in the following.

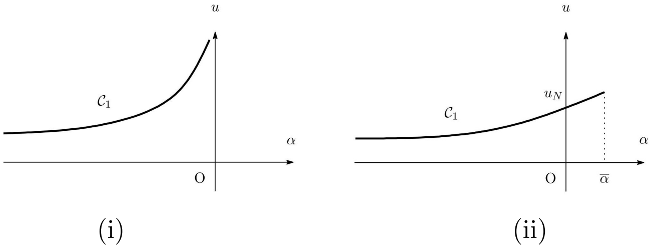


Figure: (i) $\int_\Omega a \geq 0$; (ii) $\int_\Omega a < 0$.

- $\mathcal{C}_1 = \{(\alpha, u_1(\alpha)) \in (-\infty, 0) \times W^{2,r}(\Omega)\}$ is C^∞ and increasing, and extended to $\alpha > 0$ small if $\int_\Omega a < 0$.
- $u_1(\alpha) \to u_{-\infty}$ in $H^1(\Omega)$, $\alpha \to -\infty$.

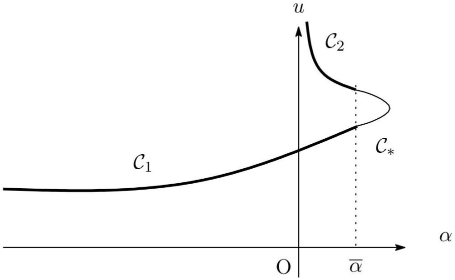
定理 2 ($\alpha > 0$)

Assume $(A.1)$, $\int_\Omega a < 0$ and $q \in \mathcal{A}_{-\infty}$. In addition to $\mathcal{C}_1$, (P_α) has a C^∞ curve $\mathcal{C}_2 = \{(\alpha, u_2(\alpha)) \in (0, \overline{\alpha}) \times W^{2,r}(\Omega)\}$ of solutions $\gg 0$. Moreover,

- (P_α) has *exactly two* nontrivial solutions $u_1(\alpha), u_2(\alpha) \gg 0$ for $\alpha \in (0, \overline{\alpha})$.
- Assume additionally $(A.2)$ and

$$(A.3) \qquad \begin{cases} a \geq 0, \quad a \not\equiv 0 \text{ in a smooth domain } D \subset \Omega \\ \text{with the condition } |\partial D \cap \partial \Omega| > 0. \end{cases}$$

Then, (P_α) has a component $\mathcal{C}_*$ of solutions $\gg 0$ including $\mathcal{C}_1$ and $\mathcal{C}_2$:



多重性

- The first solution $u_1(\alpha)$:

$$(P_\alpha) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ \partial_\nu u = \alpha u & \text{on} \quad \partial\Omega. \end{cases}$$

solution $(\alpha, u) = (0, u_0)$ with $u_0 \gg 0$ is unique and non-degenerate.

- The second solution $u_2(\alpha) := \alpha^{-\frac{1}{1-q}} w$,

$$(Q_\alpha) \qquad \begin{cases} -\Delta w = \alpha a(x)w^q & \text{in} \quad \Omega, \\ w \geq 0 & \text{in} \quad \Omega, \\ \partial_\nu w = \alpha w & \text{on} \quad \partial\Omega. \end{cases}$$

From $(\alpha, w) = (0, c_a)$, we have a unique bifurcating solution $w(\alpha) \gg 0$.

$$c_a := \left(\frac{-\int_\Omega a}{|\partial\Omega|} \right)^{\frac{1}{1-q}} > 0.$$

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