The effect of a nonlinear boundary condition with an indefinite weight on the positive solution set of the logistic elliptic equation

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Problem. Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a smooth bounded domain. Consider

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda(m(x)u - a(x)|u|^{p-2}u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}} = \lambda b(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

Here, $\lambda \in \mathbb{R}$ is a parameter, $m, a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$, $a > 0$ in Ω, m, b may change sign, $m^+ \not\equiv 0$, and

$$1 < q < 2 < p \quad \text{(i.e., } \; q - 1 < 1 < p - 1\text{).}$$

In this talk, variational methods together with a bifurcation technique are used to study the structure of the set of positive solutions for $\lambda \in \mathbb{R}$.

Regularity and positivity.

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda(m(x)u - a(x)|u|^{p-2}u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}} = \lambda b(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

If $u \in H^1(\Omega)$ is a weak solution of (P_λ), then we have

$$u \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C^\theta(\overline{\Omega}) \quad \text{with } r > N \quad \text{and } 0 < \theta < 1,$$

(Rossi '05).

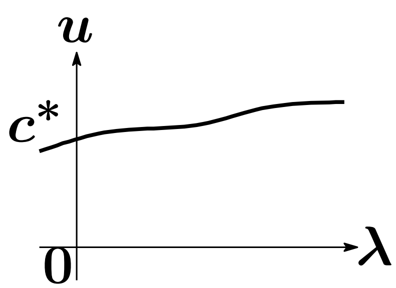
A nontrivial nonnegative solution is positive in Ω by the weak maximum principle (Gilbarg and Trudinger '83), and however, it would be difficult to deduce it is positive in the closure $\overline{\Omega}$.

Case $b^- \equiv 0$. If $b \geq 0$ then positive solution $u > 0$ in $\overline{\Omega}$ is unique for every $\lambda > 0$ (Pao '92). Then, it has been proved the problem

$$\begin{cases} -\Delta u = \lambda u - u^{p-1} & \text{in } \Omega \\ \frac{\partial u}{\partial \mathrm{n}} = u^{q-1} & \text{on } \partial\Omega \end{cases}$$

has a unique positive solution for $\lambda \in \mathbb{R}$ (García-Melián, Morales-Rodrigo, Rossi, and Suárez '08).

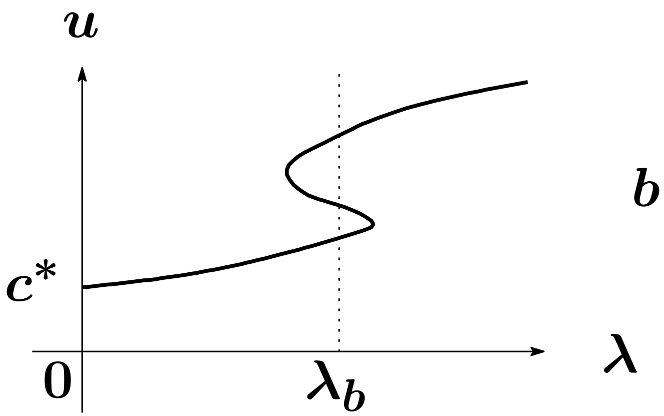
Theorem 0. Assume $b \geq 0$ and $b \not\equiv 0$. Then, (P_λ) has a unique positive solution u_λ for $\lambda > 0$, satisfying $u_\lambda \to c^*$ as $\lambda \to 0^+$.



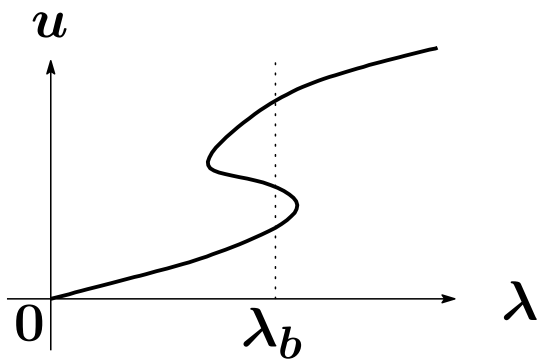
Here, $c^*(\int_{\partial\Omega} b)$ is the unique positive zero of

$$\phi(t) := t^{2-q} \int_\Omega m - t^{p-q} \int_\Omega a + \int_{\partial\Omega} b.$$

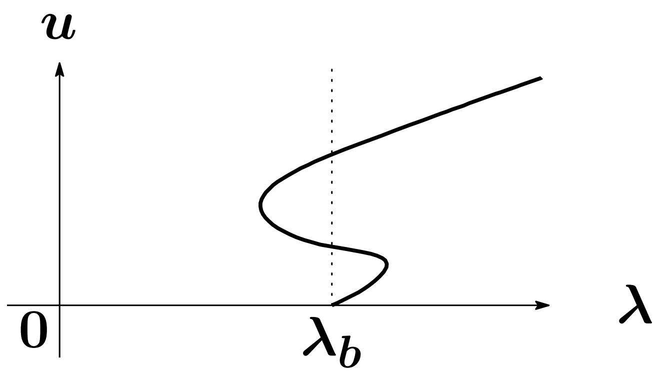
Give a *heuristic* observation if $b^- \not\equiv 0$ is considered for $\int_\Omega m < 0$.



$b^- \not\equiv 0$ and $\int_{\partial\Omega} b > 0$



$\int_{\partial\Omega} b < 0$ and $b^+ \not\equiv 0$



$b \le 0$

Constrained eigenvalue problems. Set

$$\lambda_1(m) = \inf\left\{\int_\Omega |\nabla u|^2 : u \in H^1(\Omega), \int_\Omega mu^2 = 1\right\} \left(= \int_\Omega |\nabla \varphi_1|^2\right),$$

$$\lambda_b = \inf\left\{\int_\Omega |\nabla u|^2 : u \in H^1(\Omega), \int_\Omega mu^2 = 1, \quad \int_{\partial\Omega} b|u|^q \geq 0\right\}$$

Remark. $\lambda_1(m) \leq \lambda_b \leq \lambda_1^D(m)$ ($\lambda_1^D(m)$ denotes the positive principal eigenvalue under the Dirichlet condition), and

$$\lambda_b = \begin{cases} \lambda_1(m), & b \geq 0, \\ \lambda_1^D(m), & b < 0. \end{cases}$$

Main results.

Theorem 1(Existence and multiplicity for $\lambda > 0$). Let $p \leq \frac{2N}{N-2}$ if $N > 2$. Assume $b^+ \not\equiv 0$. Then, (P_λ) has at least *one* nontrivial nonnegative solution for every $\lambda > 0$. Additionally assume

$$\int_\Omega m < 0 \quad \text{and} \quad \int_{\partial\Omega} b\varphi_1^q < 0.$$

Then:

(1) $0 < \lambda_1(m) < \lambda_b$, and

(2) if $\|a\|_\infty$ is small, then there exists $\lambda^*(a) \in (\lambda_1(m), \lambda_b)$ such that (P_λ) has at least *three* nontrivial nonnegative solutions for $\lambda \in (\lambda^*, \lambda_b)$.

Main results (continued).

Theorem 2(Uniqueness for $\lambda > 0$ close to 0). If $\int_\Omega m < 0$ then (P_λ) has *at most* one nontrivial nonnegative solution for any $\lambda \in (0, \lambda_1(m))$. The unique positive solution, if it exists, converges to c^* as $\lambda \to 0^+$.

(cf. Theorem 1.3, Morales-Rodrigo and Suárez '06)

Theorem 3(Smooth curve in $\lambda \simeq 0$). Let $m, a \in C^\theta(\overline{\Omega})$ and $b \in C^{1+\theta}(\partial\Omega)$ be assumed. If $\int_{\partial\Omega} b > 0$ then there exists a classical positive solution $u_\lambda \in C^{2+\theta}(\overline{\Omega})$ of (P_λ) for $\lambda \in (-\overline{\lambda}, \overline{\lambda})$ with some $\overline{\lambda} > 0$ such that u_λ is continuous in $C^{2+\theta}(\overline{\Omega})$ for λ, and $u_0 = c^*$.

Moreover, there is no other classical positive solution which converges to a positive constant in $C(\overline{\Omega})$ as $\lambda \to 0$.

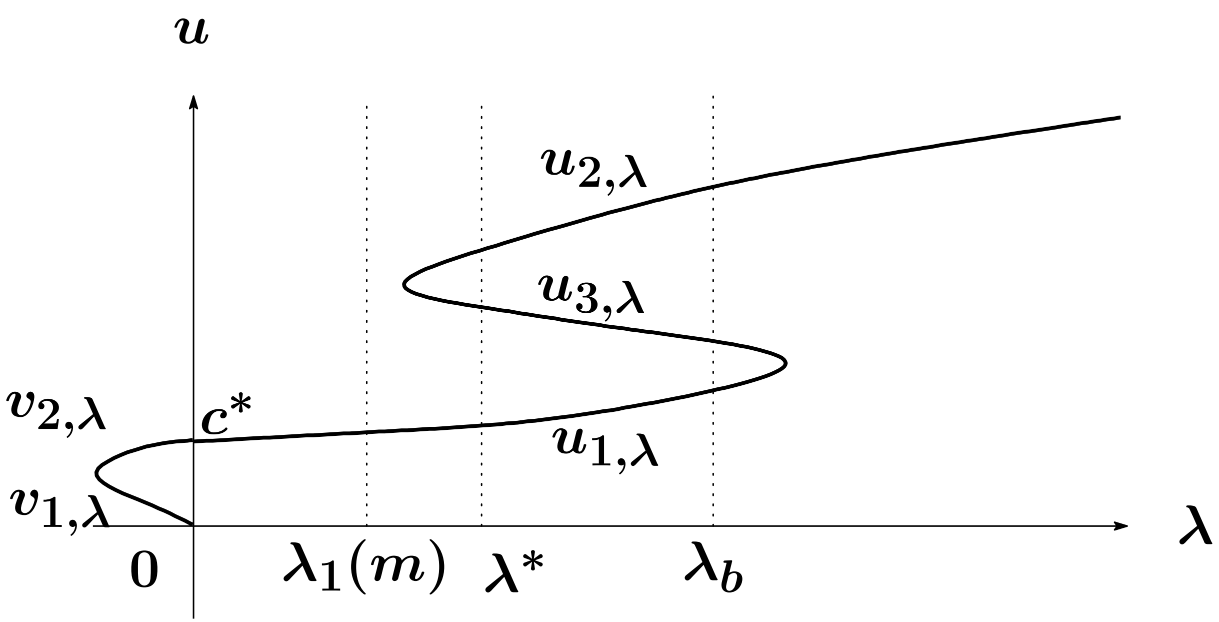
Main results (continued).

Theorem 4(Multiplicity for $\lambda < 0$). Assume $p < \frac{2N}{N-2}$ if $N > 2$. If $b^- \not\equiv 0$ and $\int_{\partial\Omega} b > 0$ then (P_λ) has at least *two* nontrivial nonnegative solutions v_1, v_2 for $\lambda \in (-\overline{\lambda}, 0)$ with some $\overline{\lambda} > 0$, satisfying

$$\begin{cases} v_1 \longrightarrow 0 \\ v_2 \longrightarrow c^* \end{cases} \quad \text{in } C^\theta(\overline{\Omega}) \quad \text{as } \lambda \to 0^-.$$

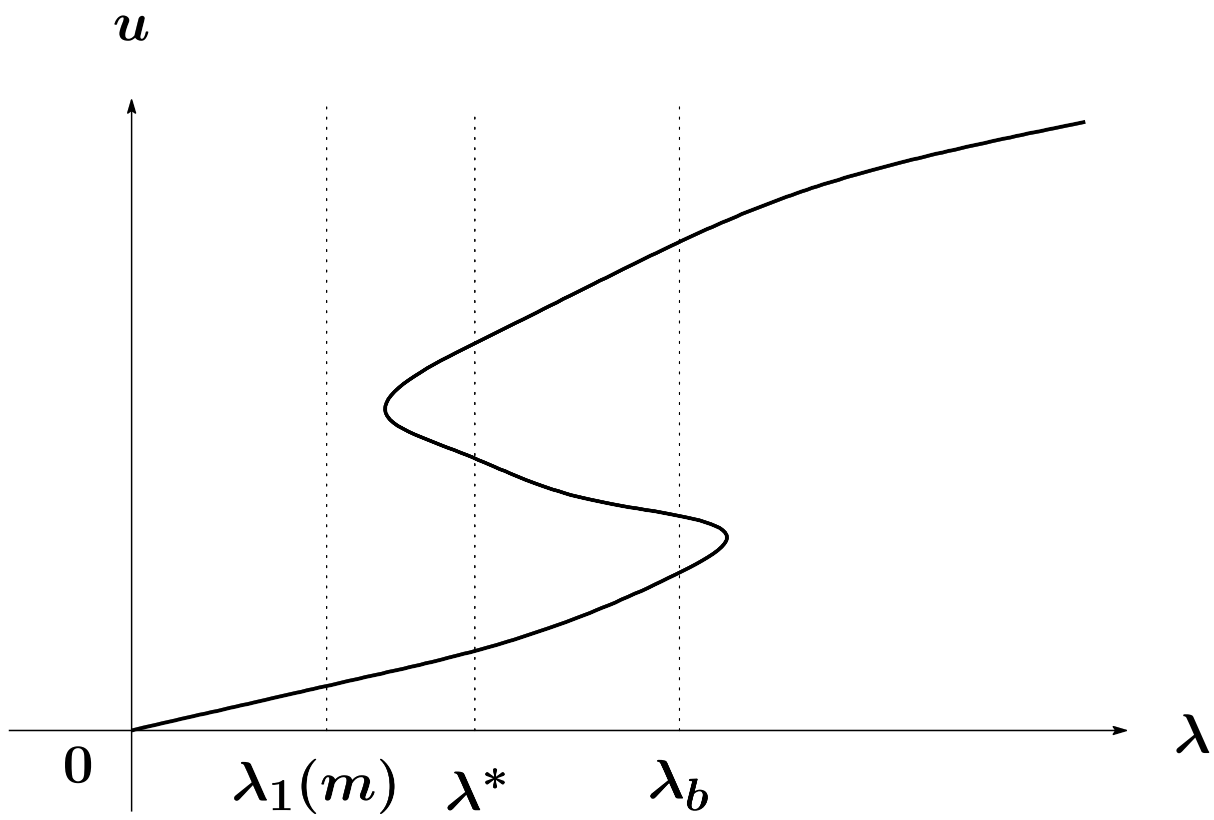
Theorem 5(Nonexistence for $\lambda \ll 0$). If m changes sign then there is no nontrivial nonnegative solution of (P_λ) for any $\lambda < 0$ sufficiently large.

"*CS* -shaped" bifurcation diagram (an expectation)



Case $\int_\Omega m < 0 < \int_{\partial\Omega} b$, **and** $\int_{\partial\Omega} b\varphi^q < 0 < \int_\Omega a\varphi_1^p \ll 1$

“S -shaped” bifurcation diagram (an expectation)



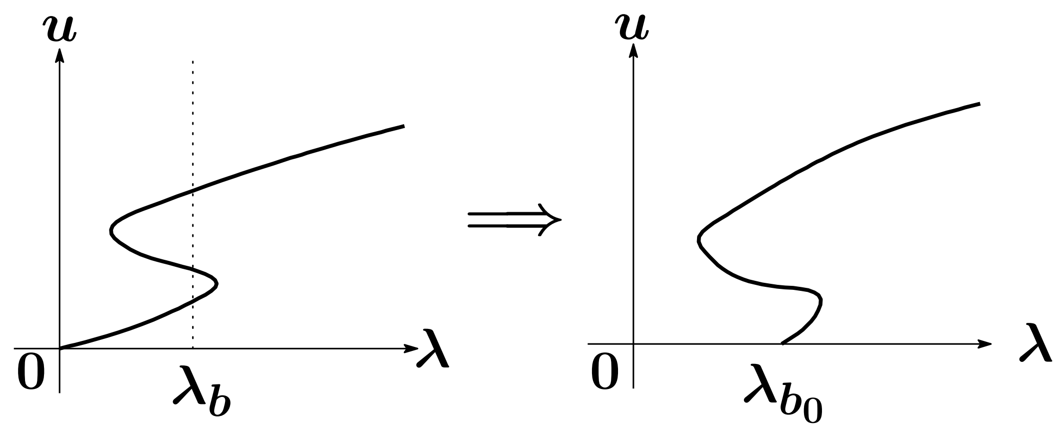
Case $\int_\Omega m < 0$, $\int_{\partial\Omega} b \le 0$, $b^+ \not\equiv 0$, **and** $\int_{\partial\Omega} b\varphi_1^q < 0 < \int_\Omega a\varphi_1^p \ll 1$

Futher problems.

(1) For the case $b_0(x) \le 0$, consider bifurcation from the zero solution and its global behavior. It can be verified that given $K > 0$ and $0 < \lambda^* < \lambda_b$,

$$\|u_{1,\lambda}\| \le C\,(\,\lambda\|b^+\|_\infty\,)^{\frac{1}{2-q}}$$

for $b(x) \le K$, $b^+ \not\equiv 0$, and $\lambda \in (0, \lambda^*)$.



"an expectation" if $b \longrightarrow b_0 \le 0$

(2) Consider the case $a(x)$ changes sign. This means the consideration of the case of superlinear nonlinearity with indefinite weight $a(x)$ in Ω and sublinear nonlinearity with indefinite weight $b(x)$ on $\partial\Omega$.

Sketch of Proof of Theorem 1. For $u \in H^1(\Omega)$, set

$$I_\lambda(u) = \frac{1}{2}E_\lambda(u) + \frac{\lambda}{p}A(u) - \frac{\lambda}{q}B(u),$$

where

$$E_\lambda(u) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega mu^2, \quad A(u) = \int_\Omega a|u|^p, \quad B(u) = \int_{\partial\Omega} b|u|^q.$$

$$\Longrightarrow \; I_\lambda \text{ is coercive for } \lambda > 0.$$

Show the existence of at least *three* nontrivial nonnegative solutions for some range of λ.

- $\exists\, u_{1,\lambda} \in B^+$ s.t. $I_\lambda(u_{1,\lambda}) = \inf_{u \in B^+} I_\lambda(u) < 0$, where $B^+ = \{u \in H^1(\Omega) : B(u) > 0\}$, since $b^+ \not\equiv 0$.

- $\exists\, u_{2,\lambda} \in E_\lambda^-$ s.t. $I_\lambda(u_{2,\lambda}) = \inf_{u \in E_\lambda^-} I_\lambda(u)$, where $E_\lambda^- = \{u \in H^1(\Omega) : E_\lambda(u) < 0\}$, if λ satisfies

$$\lambda \in (0, \lambda_b) \quad \text{and} \quad \inf_{u \in E_\lambda^-} I_\lambda(u) < 0.$$

• $\boldsymbol{\lambda} \in (0, \boldsymbol{\lambda}_b) \Longrightarrow B^+ \cap E_{\boldsymbol{\lambda}}^- = \emptyset \quad \Longrightarrow u_{1,\boldsymbol{\lambda}} \neq u_{2,\boldsymbol{\lambda}}$

($\because$ If $\boldsymbol{\lambda} \in (0, \boldsymbol{\lambda}_b)$ then

$$E_{\boldsymbol{\lambda}}(u) = \int_\Omega |\nabla u|^2 - \boldsymbol{\lambda} \int_\Omega mu^2 > 0, \quad \text{for} \quad \int_\Omega mu^2 > 0, \quad B(u) \geq 0.)$$

• $\Gamma_{\boldsymbol{\lambda}} = \{\gamma(\cdot) \in C([0,1], H^1(\Omega)) : \gamma(0) = u_{1,\boldsymbol{\lambda}},\ \gamma(1) = u_{2,\boldsymbol{\lambda}}\}$

$$c_{\boldsymbol{\lambda}} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\boldsymbol{\lambda}}(\gamma(t)) > 0.$$

$p < \dfrac{2N}{N-2} \Longrightarrow$ **compactness argument** $\Longrightarrow$ **(PS) condition**

$p = \dfrac{2N}{N-2} \Longrightarrow$ **Fatou lemma, Brezis-Lieb lemma** $\Longrightarrow$ **(PS) condition**

$\Longrightarrow$ **a variant of the Mountain Pass Theorem** $\Longrightarrow$ **the third solution** $u_{3,\boldsymbol{\lambda}}$

How to verify $\inf_{u \in E_\lambda^-} I_\lambda(u) < 0$?

By definition of λ_b, we remark

$$B(\varphi_1) < 0 \iff \lambda_1(m) < \lambda_b$$

under $\int_\Omega m < 0$. Note

$$E_\lambda(t\varphi_1) = t^2(\lambda_1(m) - \lambda) \int_\Omega m\varphi_1^2 < 0 \quad \text{for} \quad \lambda > \lambda_1(m).$$

Consider sufficient conditions for getting the inequality

$$\begin{aligned} \psi(t) &= \frac{I_\lambda(t\varphi_1)}{t^q} \\ &= -\frac{\lambda}{q} B(\varphi_1) + \frac{t^{2-q}}{2}(\lambda_1(m) - \lambda) + \frac{\lambda t^{p-q}}{p} A(\varphi_1) < 0. \end{aligned}$$

We see ψ has the global minimum

$$\psi(t_0) = -\frac{\lambda}{q}\left(B(\varphi_1) + C_{pq}\frac{\left(1 - \frac{\lambda_1(m)}{\lambda}\right)^{\frac{p-q}{p-2}}}{A(\varphi_1)^{\frac{2-q}{p-2}}}\right),$$

and thus, we are reduced to consider when $\psi(t_0) < 0$.

Note

$$\psi(t_0) < 0 \iff \left(1 - \frac{\lambda_1(m)}{\lambda}\right)^{\frac{p-q}{p-2}} > C_{pq}^{-1}(-B(\varphi_1))A(\varphi_1)^{\frac{2-q}{p-2}}.$$

If $\|a\|_\infty$ is small enough, then there exists $\lambda^* \in (\lambda_1(m), \lambda_b)$ such that $\psi(t_0) < 0$ for $\lambda \in (\lambda^*, \lambda_b)$. Hence, we obtain

$$\inf_{u \in E_\lambda^-} I_\lambda(u) < 0 \quad \text{for} \quad \lambda \in (\lambda^*, \lambda_b).$$

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Thank you for your attention.