Bifurcation analysis for indefinite weight boundary value problems with nonlinear boundary conditions

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Our problem

- $\Omega \subset \mathbb{R}^N$, $N \geq 2$, bounded domain with smooth boundary $\partial\Omega$,

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda b(x) u^p & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \geq 0$, $m \in C^\theta(\overline{\Omega})$, $\underline{\exists x_0 \in \Omega \text{ s.t. } m(x_0) > 0}$, $b \in C^{1+\theta}(\partial\Omega)$, $p > 1$.

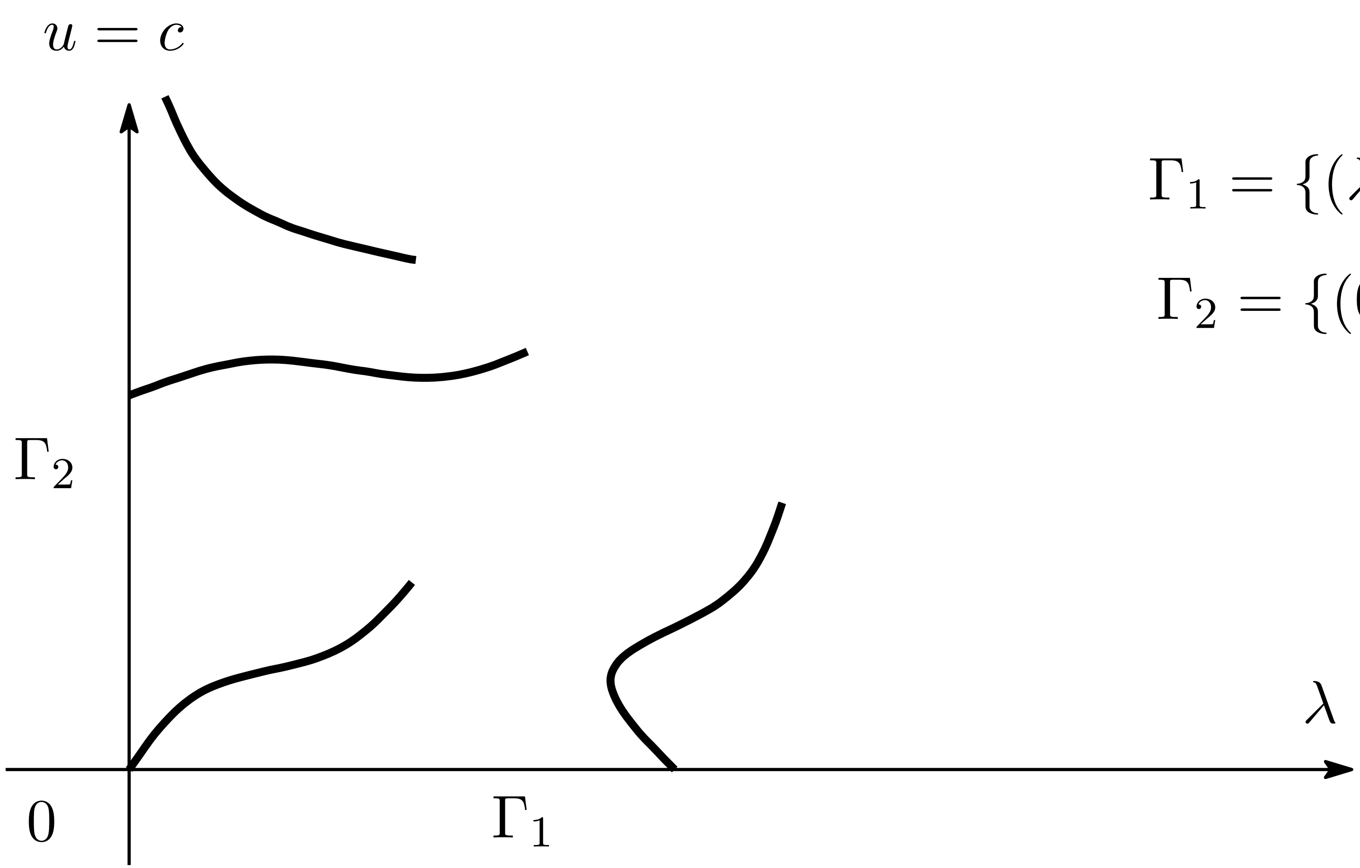
- Existence of positive solutions (λ, u)
- Initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \nabla \cdot \lambda^{-1} \nabla u + m(x)u - u^2, & (t,x) \in (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) \geq 0, & x \in \Omega, \\ (\lambda^{-1} \nabla u) \cdot \boldsymbol{n} = b(x) u^p, & (t,x) \in (0,\infty) \times \partial\Omega. \end{cases}$$

Expected bifurcation

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda b(x)u^p & \text{on } \partial\Omega, \end{cases}$$

- It is clear that our problem has two lines of trivial solutions.



$\Gamma_1 = \{(\lambda, 0)\}$

$\Gamma_2 = \{(0, c) : \text{constant } c \geq 0\}$

Our approach to bifurcation

(1) Reduce our problem to a bifurcation equation in $\mathbb{R}^2$ for bifurcation from $\Gamma_2 = \{(0, c)\}$, by using the Lyapunov-Schmidt procedure.

(2) Consider a constrained minimization problem for bifurcation from infinity, based on the first positive solution.

(3) Apply local and global bifurcation theories for bifurcation from $\Gamma_1 = \{(\lambda, 0)\}$.

(1) Bifurcation at $(\lambda, u) = (0, 0)$: Assume the condition $\int_\Omega m dx = 0$.
If $u = t + w$, $t = \frac{1}{|\Omega|}\int_\Omega u dx$, and $\int_\Omega w dx = 0$, then

$$-\Delta w + \frac{\lambda}{|\Omega|}\int_{\partial\Omega} b(t+w)^p ds = \lambda Q[\{m-(t+w)\}(t+w)] \text{ in } \Omega,$$

$$\frac{\partial w}{\partial \boldsymbol{n}} = \lambda b(t+w)^p \text{ on } \partial\Omega.$$

$\Longrightarrow$ (uniquely solvable) $w = w(\lambda, t)$ for $(\lambda, t) \simeq (0, 0)$

$\Longrightarrow$ (bifurcation equation) For $(\lambda, t) \simeq (0, 0)$,

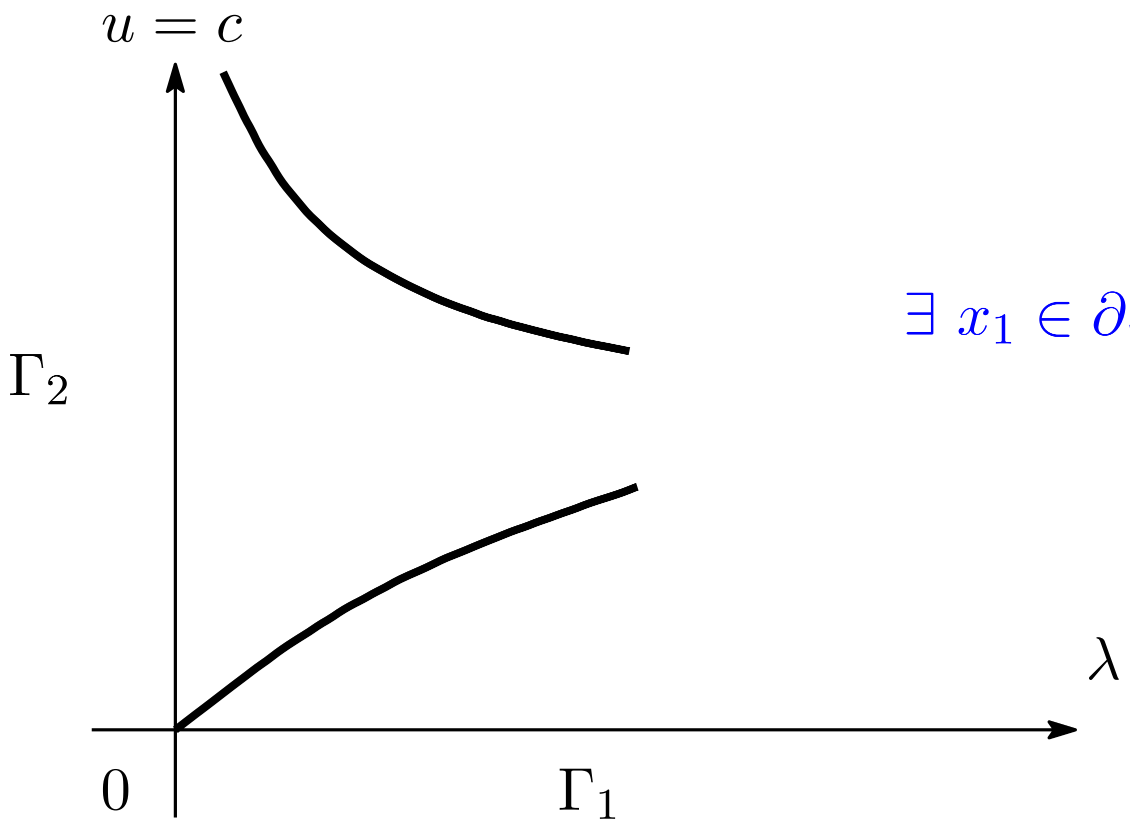
$$0 = \Phi(\lambda, u(\lambda, t)) = \int_\Omega (m - u(\lambda, t))u(\lambda, t)dx + \int_{\partial\Omega} bu(\lambda, t)^p ds$$

$$= \begin{cases} \exists\, C_1\lambda t - t^2\left(|\Omega| - \int_{\partial\Omega} bds \right) + \text{ higher order terms}, & p = 2 \\ \exists\, C_1\lambda t - t^2|\Omega| + \text{ higher order terms}, & p = 3, 4, 5, \ldots \end{cases}$$

(2) Variational positive solution

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda b(x) u^2 & \text{on } \partial\Omega, \end{cases}$$

Case: $\int_\Omega m dx = 0, \quad p = 2, \quad$ and $\quad |\Omega| > \int_{\partial\Omega} b ds$



$N = 2, 3$

$\exists\ x_1 \in \partial\Omega$ s.t. $b(x_1) > 0$

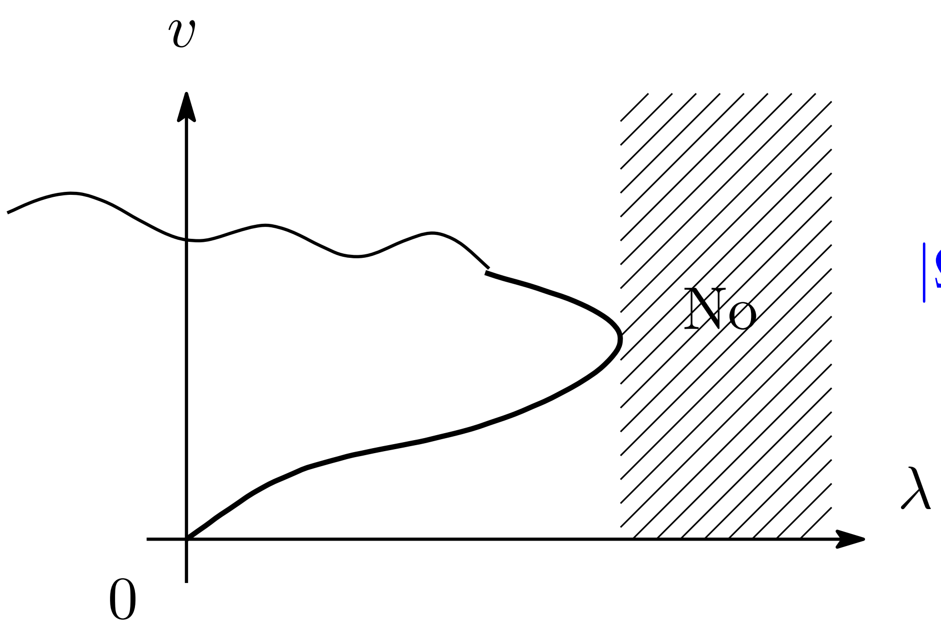
(U. (2004))

García-Melian, Morales-Rodrigo, Rossi, and Suárez (2008) considered the similar problem

$$\begin{cases} -\Delta v = \lambda v - v^p & \text{in} \ \ \Omega, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = v^r & \text{on} \ \ \partial\Omega, \qquad p, r > 0. \end{cases}$$

If (λ, u) satisfies our boundary value problem, then

$$v = \lambda u, \ \ m = 1, \ \ b = 1 \implies p = r = 2$$



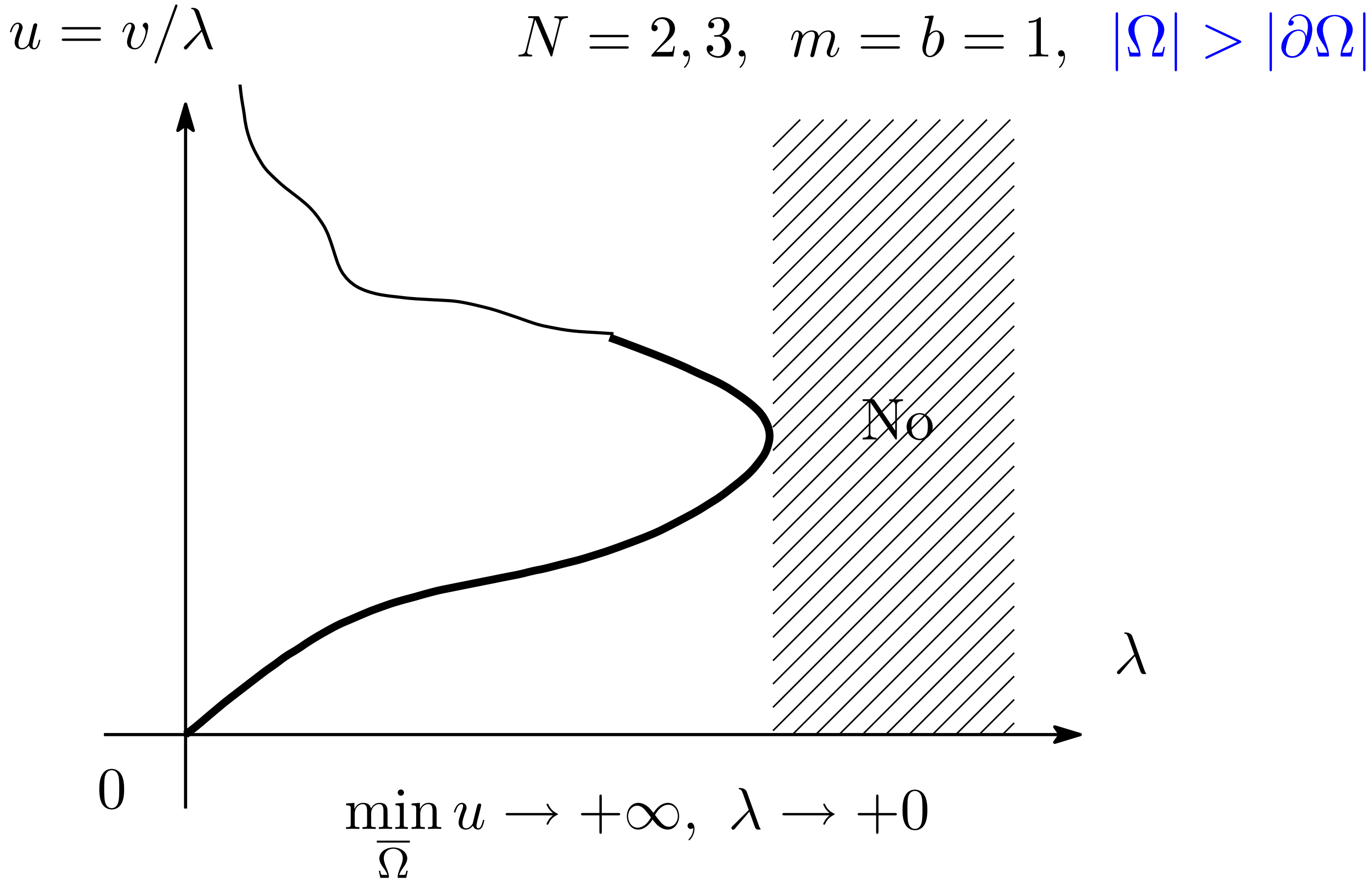
$N = 2, 3$

$|\Omega| > |\partial\Omega|$

Convert to our problem

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \ \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda b(x) u^2 & \text{on } \ \partial\Omega, \end{cases}$$

and then



(3) Bifurcation from the λ-axis Γ_1: Study

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda b(x) u^p & \text{on } \partial\Omega, \end{cases}$$

with $\int_\Omega m dx < 0$. If we consider the linearized eigenvalue problem at $u = 0$:

$$\begin{cases} -\Delta \varphi = \lambda m(x) \varphi & \text{in } \Omega, \\ \dfrac{\partial \varphi}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

there is a unique positive principal eigenvalue $\lambda_1(m)$ (Brown and Lin (1980)). Hence, a bifurcation point is unique in $\lambda > 0$.

In the case $\int_\Omega m dx \geq 0$, there is no positive principal eigenvalue.

Study bifurcation for a general class of nonlinear boundary value problems:

$$\begin{cases} -\Delta u = \lambda(m(x)u + g_1(x,u)) & \text{in } \ \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda(\sigma(x)u + g_2(x,u)) & \text{on } \ \partial\Omega, \end{cases}$$

with $\int_\Omega m dx + \int_{\partial\Omega} \sigma ds < 0$, where

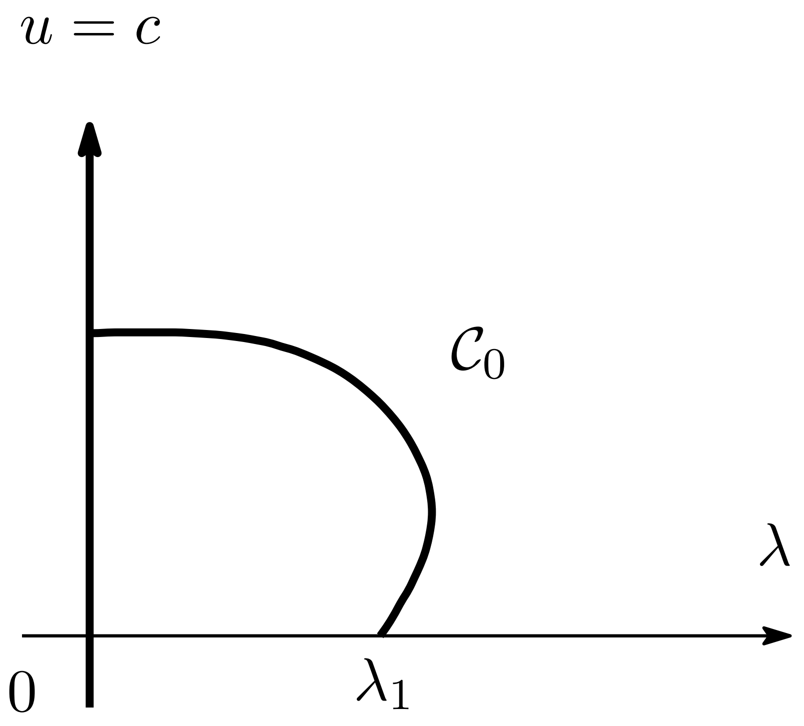
$$\begin{cases} \lim\limits_{u \to +0} \dfrac{g_1(x,u)}{u} = 0 & \text{uniformly in } \overline{\Omega}, \\ \lim\limits_{u \to +0} \dfrac{g_2(x,u)}{u} = 0 & \text{uniformly on } \partial\Omega. \end{cases}$$

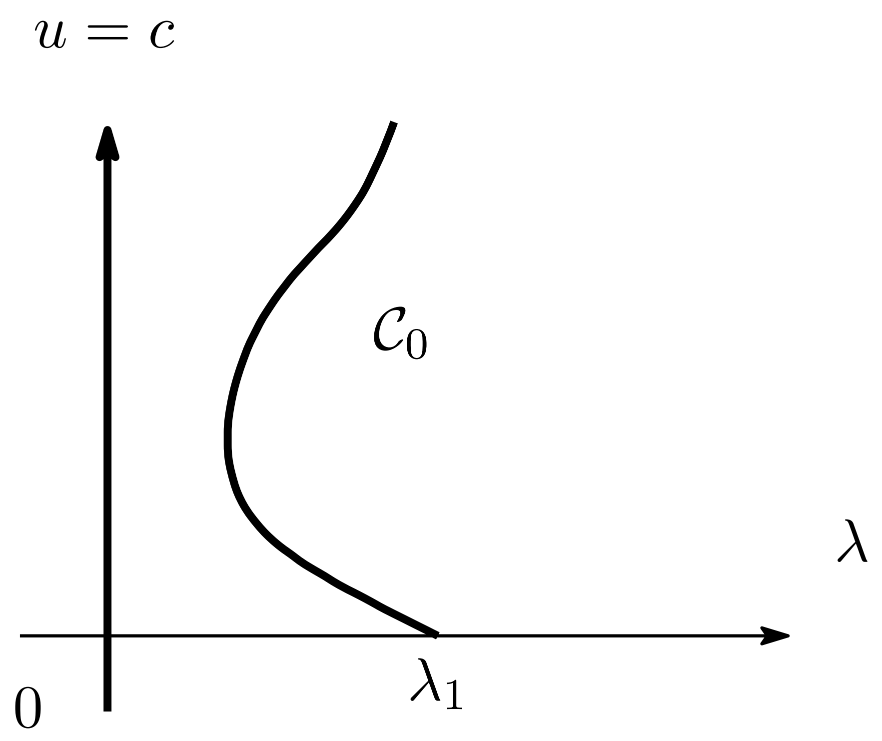
There is a unique positive principal eigenvalue $\lambda_1(m,\sigma)$ of

$$\begin{cases} -\Delta \varphi = \lambda m(x) \varphi & \text{in } \ \Omega, \\ \dfrac{\partial \varphi}{\partial \boldsymbol{n}} = \lambda \sigma(x) \varphi & \text{on } \ \partial\Omega. \end{cases} \qquad \text{(U. (2006))}$$

Put $\quad \mathcal{S} := \{(\lambda, u) \in [0, \infty) \times C(\overline{\Omega}) : \quad u \text{ is a positive solution for } \lambda\}.$

Theorem. *$\overline{\mathcal{S}}$ contains a subcontinuum $\mathcal{C}_0$ of positive solutions emanating from Γ_1 at $(\lambda_1, 0)$, which is unbounded.*



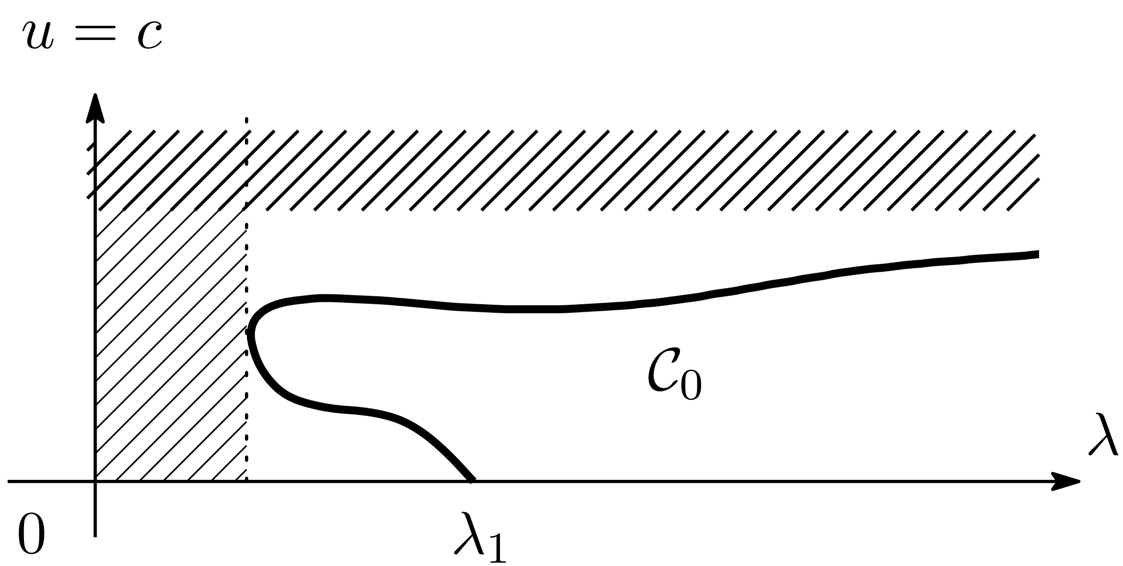


Case of no secondary bifurcation from Γ_2

For the behavior of bifurcation component, we consider $2 \leq N \leq 5$, $1 < p < p_0(N)$, $p_0(N) = 1 + \frac{4}{N+3+\sqrt{N^2-2N+25}} (< \frac{3}{2})$, and the problem

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda b(x) u^p & \text{on } \partial\Omega. \end{cases}$$

If $\overline{m} = \dfrac{1}{|\Omega|} \int_\Omega m dx$, $m_p = \dfrac{(p-1)^{p-1}(2-p)^{2-p}}{(-\overline{m})^{2-p}}$, $|\Omega| > m_p \int_{\partial\Omega} b ds$, and $\int_{\partial\Omega} b\, \varphi_1^{p+1} ds > 0$, then we have



For obtaining such subcontinuum $\mathcal{C}_0$, we prove that:

(a) there is a unbounded subcontinuua of positive solutions bifurcating from $(\lambda_1, 0)$, by applying the unilateral global bifurcation theory proposed by López-Gómez (2001),

(b) the bifurcation from $(\lambda_1, 0)$ is to the left, from the condition that $\int_{\partial\Omega} b\,\varphi_1^{p+1} ds > 0$,

(c) there is no bifurcation from Γ_2, from the condition that $|\Omega| > m_p \int_{\partial\Omega} b ds$,

(d) for any $\Lambda > 0$, there is a constant $C > 0$ such that a positive solution u for $0 < \lambda \leq \Lambda$ satisfies the condition $\|u\|_{C(\overline{\Omega})} \leq C$, by virtue of $1 < p < p_0(N)$.

Proof of the *a priori* upper bound: Let $\Lambda_0 > 0$, and $0 < \lambda \leq \Lambda_0$.

- Calculation of $\int_\Omega -\Delta u \cdot u dx$:

$$\int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega m u^2 dx - \lambda \int_\Omega u^3 dx + \lambda \int_{\partial\Omega} h u^{p+1} ds \qquad (1)$$

- Use of the Filo and Kačur inequality (1995): $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ s.t.

$$\int_{\partial\Omega} |u|^{p+1} ds \leq \varepsilon \|u\|_{1,2}^2 + C_\varepsilon \left(\int_\Omega |u|^{p+1} dx \right)^r, \quad \forall u \in W^{1,2}(\Omega), \qquad (2)$$

where $1 < p < \frac{N+1}{N-1}$ and $r > \frac{N-p(N-2)}{N+1-p(N-1)}$

$$\Longrightarrow \|u\|_{1,2}^2 \leq C \int_\Omega u^2 dx + \lambda \left\{ C \left(\int_\Omega u^3 dx \right)^{\frac{(p+1)r}{3}} - \int_\Omega u^3 dx \right\} \qquad (3)$$

$$\frac{(p+1)r}{3} < 1 \implies \|u\|^2_{W^{1,2}(\Omega)} \leq C\left(1 + \int_\Omega u^2 dx\right) \tag{4}$$

- Calculation of $\int_\Omega -\Delta u dx$:

$$\int_\Omega u^2 dx = \int_\Omega mu dx + \int_{\partial\Omega} hu^p ds, \quad 1 < p < 2 \tag{5}$$

$$\implies \int_\Omega u^2 dx \leq C\left\{\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} + \left(\int_{\partial\Omega} u^2 ds\right)^{\frac{p}{2}}\right\} \tag{6}$$

- Use of the Afrouzi and Brown inequality (1999):

$$\int_{\partial\Omega} u^2 ds \leq \int_\Omega |\nabla u|^2 dx + \exists C' \int_\Omega u^2 dx, \quad \forall u \in W^{1,2}(\Omega). \tag{7}$$

$$\implies \int_\Omega u^2 dx \leq C\left\{\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} + \left(1 + \int_\Omega u^2 dx\right)^{\frac{p}{2}}\right\}$$

Remarks on the *a priori* upper bound

- The *a priori* upper bound breaks down at $\lambda = 0$.

- When $p > \frac{3}{2}$, García-Melian, Morales-Rodrigo, Rossi, and Suárez (2008) established the same kind of *a priori* bounds for positive solutions, while we consider the case $1 < p < p_0(N)(< \frac{3}{2})$.

Thank you for your attention.