Existence of a loop of positive solutions for concave-convex problems with indefinite weights

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CT22, Equadiff 2019
Leiden, The Netherlands
11 July 2019

Problem

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a smooth bounded domain. Consider the nontrivial solutions of (P_λ) with $\lambda \in \mathbb{R}$:

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda a(x) u^q + b(x) u^p & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

- $a, b \in C(\overline{\Omega})$, a changes sign, and $b > 0$ somewhere;
- $0 < q < 1 < p < \frac{N+2}{N-2}$ (concave-convex)
- $\partial_\nu = \frac{\partial}{\partial \nu}$, where ν is the unit exterior normal to $\partial\Omega$.

A *solution* u of (P_λ) if $u \in W^{2,r}(\Omega)$, $r > N$, satisfies (P_λ) (so $u \in C^1(\overline{\Omega})$.)

Previous works

- Ambrosetti, Brezis and Cerami (1994)

  ▷ $\boxed{a,\ b \equiv 1}$, Dirichlet, existence and multiplicity

- de Figueiredo, Gossez and Ubilla (2003)

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- Tarfulea (1998)

  ▷ $\boxed{a \text{ changes sign},\ \ b \equiv 1}$, Neumann, local existence for $\lambda > 0$ small

- Alama (1999)

  ▷ $\boxed{a \equiv 1,\ \ b \text{ changes sign}}$, Neumann, existence and multiplicity

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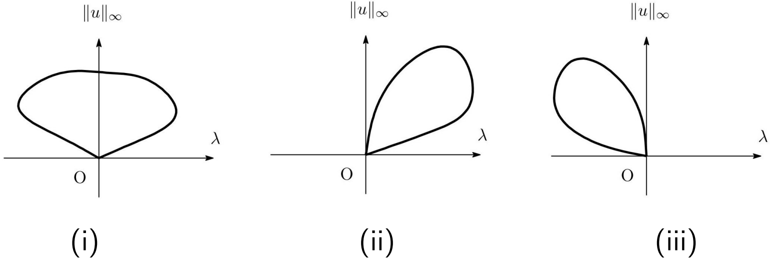

Figure: (i) $\int_\Omega b < 0$; (ii) $\int_\Omega b \geq 0$, $\int_\Omega a < 0$; (iii) $\int_\Omega b \geq 0$, $\int_\Omega a > 0$.

Remark. $(-\lambda)(-a) = \lambda a$.

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  √ The strong maximum principle (SMP) is *not* applicable, since the term $a(x)t^q$ does not satisfy the slope condition. So, a nontrivial solution is not necessarily positive in Ω.

Assumptions

Assume some certain conditions for getting a *bounded subcontinuum* of nontrivial solutions of (P_λ). Let

$$\Omega_\pm^\psi := \{x \in \Omega : \psi \gtrless 0\}.$$

- (H_{ab}) : $\Omega_+^a \cap \Omega_+^b \neq \emptyset$, and $\Omega_-^a \cap \Omega_+^b \neq \emptyset$. ($\Longrightarrow$ *upper bound of* $|\lambda|$)

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- (H_a) : $\Omega_{\pm}^{a}$ consist of a finite number of connected components. ($\Longrightarrow$ *no bifurcation from zero at* $\lambda \neq 0$)
- (H_b) : when $\overline{\Omega_{+}^{b}} \subset \Omega$, some growth condition of b^{+} in Ω_{+}^{b} is imposed in a tubular neighborhood of $\partial\Omega_{+}^{b}$ (cf. Amann and López-Gómez (1998)). ($\Longrightarrow$ *upper bound of the uniform norm for solutions*)

Main theorem

Under the conditions, if $\int_\Omega b < 0$ *then* (P_λ) *has a subcontinuum* $\mathcal{C}_0$ *in* $\mathbb{R} \times C^1(\overline{\Omega})$ *of solutions which satisfies*

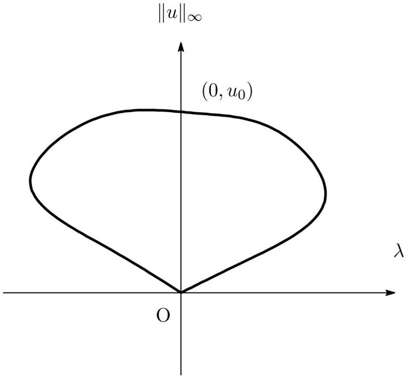
$$\mathcal{C}_0 \cap \{(\lambda, 0)\} = \{(0, 0)\}.$$

Moreover, $\mathcal{C}_0$ *is a* loop, *i.e.,...*

(i) $(0, u_0) \in \mathcal{C}_0$ for some positive solution u_0 of (P_λ) with $\lambda = 0$;

(ii) $\mathcal{C}_0$ does not contain any small positive solutions for $\lambda = 0$;

(iii) The bifurcation at $(0, 0)$ is subcritical and supercritical.



Regularization scheme

To overcome the difficulty that (P_λ) is not differentiable at $u = 0$, we consider ε-regularization of (P_λ) with $\varepsilon > 0$:

$$(P_{\lambda,\varepsilon}) \qquad \begin{cases} -\Delta u = \lambda a(x)(u+\varepsilon)^{q-1}u + b(x)u^p & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

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$\{(\lambda, 0)\}$ still remains as a trivial line $u = 0$. Linearize $(P_{\lambda,\varepsilon})$ at $u = 0$ as

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which possesses $\lambda_{1,\varepsilon}^- = 0, \quad \lambda_{1,\varepsilon}^+ > 0$, exactly two principal eigenvalues, and $\lambda_{1,\varepsilon}^+ \to 0$ as $\varepsilon \searrow 0$.

Case $\int_\Omega b < 0$

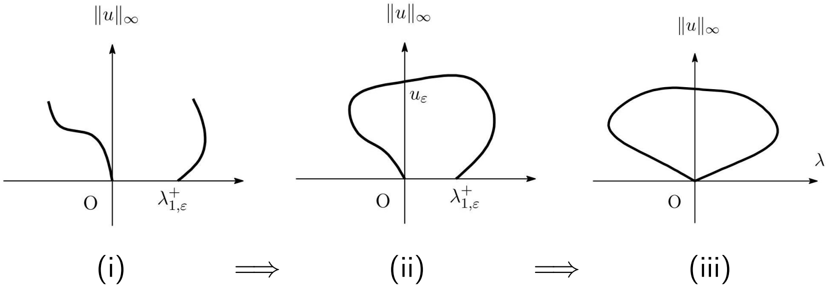

Figure: (i) local bifurcation ; (ii) global bifurcation (the a priori bounds) ; (iii) Whyburn's topological approach ($\varepsilon \searrow 0$)

No bifurcation from $(\lambda, 0)$ at $\lambda \neq 0$

Proposition. Assume (H_a). Then we have no bifurcation from zero at any $\lambda \neq 0$.

$(H_a): \ \Omega^a_\pm = \{x \in \Omega : a(x) \gtrless 0\}$ consist of a finite number of connected components.

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Proof. By contradiction we assume $\lambda_n \to \lambda_0 > 0$ and $\|u_n\|_{H^1(\Omega)} \to 0$. Then we deduce, up to a subsequence,

$$\text{either} \qquad \int_\Omega a(x) u_n^{q+1} \leq 0, \qquad \text{or} \qquad u_n \not\equiv 0 \text{ in } \Omega_+^a.$$

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If $u_n \not\equiv 0$ in Ω_+^a then there may exist a connected open subset Ω' of Ω_+^a such that $u_n \not\equiv 0$ in Ω' by (H_a), so $u_n > 0$ in Ω' by the SMP. By a comparison argument using subsolutions, it follows that

$$u_n \geq \psi \quad \text{in a ball } B \Subset \Omega',$$

where ψ is a positive eigenfunction of the smallest eigenvalue of the problem $-\Delta\psi = \lambda a(x)\psi$ in B, $\psi|_{\partial B} = 0$.

Concluding remarks

- Nonlinearities u^q and u^p can be extended to a fully concave-convex class $f(u)$ and $g(u)$, respectively.
- The Dirichlet case $u|_{\partial\Omega} = 0$ can be argued similarly. In this case, we have two principal eigenvalues $\lambda_{1,\varepsilon}^{-} < 0 < \lambda_{1,\varepsilon}^{+}$ of the linearized eigenvalue problem, and $\lambda_{1,\varepsilon}^{\pm} \to 0$.
- We don't know if any nontrivial solution of (P_λ) implies a positive solution, but when q is close to 1, *every* nontrivial solution is positive in the case $b \geq 0$.

Thank you for your kind attention.