Blowing-up behavior of principal eigenvalues for Neumann boundary conditions

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Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a smooth bounded domain.

Consider positive principal eigenvalue of the problem

$$\begin{cases} -\Delta\phi = \lambda\, g(x)\, \phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- $g \in L^\infty(\Omega)$ changes sign,
- $\lambda \in \mathbb{R}$ is an eigenvalue parameter.

Brown-Lin ('80) have proved that:

- there exists a positive principal eigenvalue $\iff$ $\int_\Omega g\, dx < 0$,
- this is unique (denoted by $\lambda_1(g)$) and simple,
- $\lambda_1(g) = \inf \left\{ \dfrac{\displaystyle\int_\Omega |\nabla v|^2\, dx}{\displaystyle\int_\Omega g v^2\, dx} : v \in W^{1,2}(\Omega)\ ,\ \displaystyle\int_\Omega g v^2\, dx > 0 \right\}$. back

Our aim is to obtain necessary and sufficient conditions for the condition

$$\lim_{j \to \infty} \lambda_1(g_j) = \infty \qquad \text{Blowing-up behavior}$$

under the assumption that

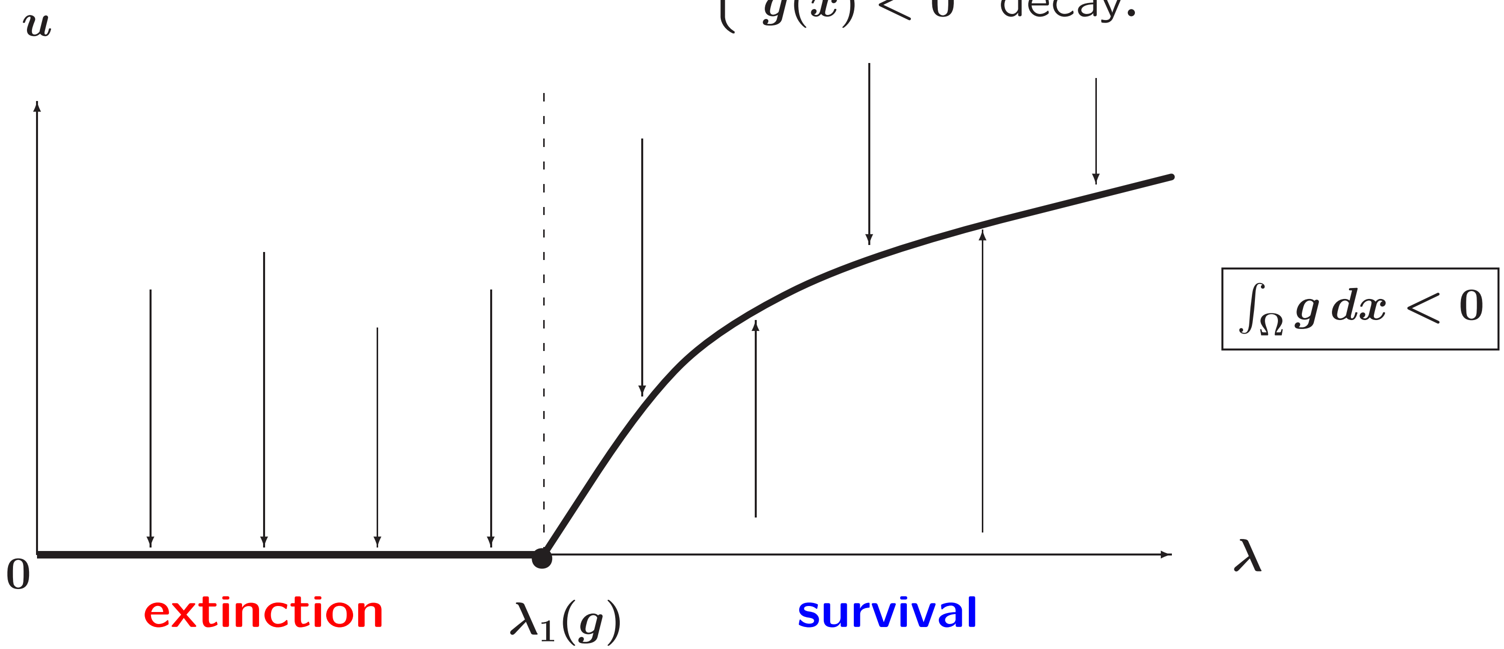
$$\sup_{j \geq 1} \|g_j\|_\infty < \infty. \quad \text{(unifromly bounded in } \Omega\text{)}$$

Our motivation for the study is from the existence of positive solutions to the semilinear problem of logistic type:

$$\begin{cases} -\Delta u = \lambda(g(x)u - u^2) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- u is the population density of some species,
- λ is a reciprocal number of the diffusion coefficient,
- $g(x)$ is the local growth or decay rate $\begin{cases} g(x) > 0 & \text{growth}, \\ g(x) < 0 & \text{decay}. \end{cases}$



Define the interval for survival of the diffusion coefficient by

$$I_g := \left(0\,,\ \frac{1}{\lambda_1(g)}\right).$$

$$\lim_{j\to\infty} \lambda_1(g_j) = \infty \iff I_{g_j} \text{ vanishes.}$$

$\iff$ which environment is worst in a given class ?

$$\lim_{j\to\infty} \|(g_j)^+\|_\infty = 0 \implies \lim_{j\to\infty} \lambda_1(g_j) = \infty\ ?$$

(Conjecture) ss

Does the local growth rate uniformly shrinking

lead to the extinction for species ?

Cantrell-Cosner ('89) have proved under the Dirichlet condition that

$$\lim_{j\to\infty} \lambda_1(g_j) = \infty$$

$$\Updownarrow$$

$$\limsup_{j\to\infty} \int_\Omega g_j \psi \, dx \le 0 \,, \quad \forall \psi \in L^1(\Omega) \text{ s.t. } \psi \ge 0 \text{ a.e. in } \Omega \,. \qquad \text{(CC)}$$

Thm1

We note that

$$\int_\Omega g_j \psi \, dx \le \int_\Omega (g_j)^+ \psi \, dx \quad \text{for} \quad \psi \ge 0.$$

where $g^+ = \max\{g, 0\}$. Therefore

$$\lim_{j\to\infty} \|(g_j)^+\|_\infty = 0 \implies \limsup_{j\to\infty} \int_\Omega g_j \psi \, dx \le 0.$$

(means " the conjecture is correct ")

For the Neumann case, the conjecture is no longer correct.

Example 1: Let $\Omega = (0,1) \subset \mathbb{R}$, and define

$$g_j(x) = \begin{cases} \dfrac{1}{j}\,, & x \in \left[0,\ 1-\frac{1}{j}\right) \\ -1\,, & x \in \left[1-\frac{1}{j},\ 1\right] \end{cases}$$ prob1

In this case, $\lambda_1(g_j)$ is bounded above. Indeed, we put

$$v_j(x) = -\frac{x}{j} + k\,, \qquad x \in [0\ ,\ 1]$$

with $0 < k < 1$. Then we note

$$\int_0^1 (v_j')^2\,dx = \frac{1}{j^2}\,,$$

$$\int_0^1 g_j(v_j)^2\,dx = \frac{k(1-k)}{j^2} + \mathrm{o}\left(\frac{1}{j^2}\right) \qquad (j \to \infty).$$

$$\therefore\quad \lambda_1(g_j) \le \frac{\displaystyle\int_0^1 (v_j')^2\,dx}{\displaystyle\int_0^1 g_j(v_j)^2\,dx} = \frac{1}{k(1-k)+\mathrm{o}(1)} \qquad (j \to \infty).$$ conjecture

Saut-Scheurer ('78) have proved under the Neumann condition that

$$\lambda_1(g) \geq \mu_2 \left(\|g^+\|_\infty + \frac{\|g\|_2^2}{\left|\int_\Omega g\,dx\right|} \right)^{-1} \quad \text{if} \quad \int_\Omega g\,dx < 0,$$

where μ_2 is the first positive eigenvalue of the problem

$$\begin{cases} -\Delta w = \mu\, w & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

From this estimate we see

$$\left.\begin{array}{l} \lim\limits_{j\to\infty} \|(g_j)^+\|_\infty = 0 \\ \lim\limits_{j\to\infty} \dfrac{\|g_j\|_2^2}{\left|\int_\Omega g_j\,dx\right|} = 0 \end{array}\right\} \Longrightarrow \lim_{j\to\infty} \lambda_1(g_j) = \infty. \qquad \text{Thm3c}$$

The following example for the blowing-up is due to Saut-Sheurer:

Example 2: Let $g \in L^\infty(\Omega)$ with $g \not\leq 0$ and $\int_\Omega g\,dx < 0$, and define

$$g_j = \sigma_j g \quad \text{with} \quad \sigma_j \downarrow 0 \qquad \text{prob2}$$

In another direction, Cantrell-Cosner gave an interesting example for the blowing-up, which is:

$$\Omega = (0, \pi) \subset \mathbb{R}\,, \qquad g_j(x) = -\sin(2j+1)x\ .$$

In this case, Thm4

$$\lim_{j\to\infty} \int_\Omega g_j \psi \, dx = 0\ , \qquad \forall \psi \geq 0.$$

We remark that

$$\|g_j\|_\infty = 1\ , \qquad \|g_j\|_2 = \sqrt{\frac{\pi}{2}}\ . \qquad \text{(means "not shrinking to } 0\text{")}$$

Dispersing foods lead to the extinction for species.

Main results

Theorem 1: Condition (CC) is also necessary in the Neumann case.

Theorem 2: Under

$$\limsup_{j\to\infty}\int_\Omega g_j\,dx < 0,$$

the condition

$$\limsup_{j\to\infty}\int_\Omega g_j\psi\,dx \le 0\,, \quad \text{not constant } \forall\psi\in L^1(\Omega) \text{ s.t. } \psi\ge 0, \text{ a.e. in } \Omega, \quad \text{(CC2)}$$

is sufficient. (means "Cantrell-Cosner's criterion remains true")

Thm3 Thm4

Case

$$\limsup_{j\to\infty} \int_\Omega g_j \, dx = 0 \qquad \text{(CC1)}$$

is critical. Thm4

Theorem 3: Under condition (CC1) we have $\lim_{j\to\infty} \lambda_1(g_j) = \infty$, provided we assume in addition to (CC2)

$$\lim_{j\to\infty} \frac{\|g_j\|^2_{(p^*)'}}{\left|\int_\Omega g_j \, dx\right|} = 0 \,, \qquad \text{SSc}$$

where $(p^*)' = p^*/(p^* - 1)$,

$$\text{(Sobolev's critical exponent)} \quad p^* = \begin{cases} \dfrac{2N}{N-2}\,, & N \geq 3, \\ \infty\,, & N = 1, 2. \end{cases}$$

Example 3 Let $\Omega = (0, 3) \subset \mathbb{R}$ and define

$$g_j(x) = \begin{cases} 1 - x^{1/j} \,, & 0 \le x \le 1, \\ \dfrac{1}{9j} x\,(x-1)(x-10) \,, & 1 < x \le 3. \end{cases}$$

Then this illustrates Theorem 3. Note that

$$\|(g_j)^+\|_\infty = 1.$$

Introduce

$$G_j \in C^1(\overline{\Omega}) \quad \begin{cases} \Delta G_j = g_j & \text{in } \Omega, \\ \dfrac{\partial G_j}{\partial n} \leq 0 & \text{on } \partial\Omega \end{cases} \qquad \left(\int_\Omega g_j\, dx < 0\right)$$

Theorem 4: Under conditions (CC2) and (CC1) we have $\lim_{j\to\infty} \lambda_1(g_j) = \infty$ if we assume the following five conditions:

$$\limsup_{j\to\infty} \left(\sup_{x\in\Omega} G_j(x)\right) \leq 0,$$

$$\limsup_{j\to\infty} \left(\operatorname{esssup}_{x\in\Omega}(-G_j(x)g_j(x))\right) \leq 0,$$

$$\sup_{j\geq 1} \frac{\|G_j g_j\|_{(p^*)'}}{\left|\int_\Omega g_j\, dx\right|} < \infty,$$

$$\sup_{j\geq 1} \frac{\int_\Omega (-G_j g_j)\, dx}{\left(\int_\Omega g_j\, dx\right)^2} < \infty,$$

$$\lim_{j\to\infty} \left\| \frac{\partial G_j}{\partial n} \right\|_{(q^*)',\, \partial\Omega} = 0 \qquad \left(q^* = \frac{2(N-1)}{N-2}\right).$$

For $g_j(x) = -\sin kx \ (k = 2j+1)$ we define

$$G_j(x) = \frac{1}{k^2} \sin kx,$$

so that

$$\frac{\int_0^\pi |G_j g_j| \, dx}{\left| \int_0^\pi g_j \, dx \right|} = \frac{\pi}{4k}, \qquad \frac{\int_0^\pi (-G_j g_j) \, dx}{\left(\int_0^\pi g_j \, dx \right)^2} = \frac{\pi}{8}.$$

Open problems

(1) Give a sufficient condition of $\{g_j\}$ for $\lambda_1(g_j)$ being bounded above in the case that $(g_j)^+$ uniformly shrinks to zero. Ex1

(2) Whether the stronger assumption that g_j uniformly shrinks to zero is sufficient for the blowing-up, or not. Ex2

Sketch of proof of Thm 1: For a contradiction we consider

$$\sup_{j\geq 1} \lambda_1(g_j) < \infty$$

by choosing subsequences if necessary. Let ϕ_j be a normalized positive eigenfunction of $\lambda_1(g_j)$ as $\int_\Omega |\nabla\phi_j|^2\,dx = 1$. It follows that

$$1 = \int_\Omega |\nabla\phi_j|^2\,dx = \lambda_1(g_j)\int_\Omega g_j\phi_j^2\,dx = \lambda_1(g_j)\;t_j^2\int_\Omega g_j\left(1+\frac{w_j}{t_j}\right)^2\,dx$$

where

$$\phi_j = t_j + w_j, \quad t_j = \frac{1}{|\Omega|}\int_\Omega \phi_j\,dx.$$

If $\|w_j/t_j\|_{W^{1,2}} \ll 1$, then

$$\limsup_{j\to\infty}\int_\Omega g_j\,dx < 0 \implies \int_\Omega g_j\,dx < 0\ (j \gg 1) \implies \int_\Omega g_j\left(1+\frac{w_j}{t_j}\right)^2\,dx < 0\ (j \gg 1)$$

Note that

$$\|w_j\|_{W^{1,2}} \simeq \|\nabla w_j\|_{L^2} = \|\nabla \phi_j\|_{L^2} = 1 \qquad \left(\because \int_\Omega w_j\, dx = 0\right)$$

$$\Longrightarrow w_j \longrightarrow \hat{w} \quad \text{weakly in } W^{1,2}(\Omega) \text{ and storngly in } L^2(\Omega)$$

If $\|w_j/t_j\|_{W^{1,2}} \geq \exists\delta$, then $|t_j|$ is bounded, so that $t_j \longrightarrow \hat{t}$. Define $\hat{\phi} = \hat{t} + \hat{w}$. It follows that

$$1 = \lambda_1(g_j) \int_\Omega g_j \phi_j^2\, dx = \lambda_1(g_j) \int_\Omega g_j(\phi_j^2 - \hat{\phi}^2)\, dx + \lambda_1(g_j) \int_\Omega g_j \hat{\phi}^2\, dx$$

$$\left(\limsup_{j\to\infty} \int_\Omega g_j \psi\, dx \leq 0, \quad \text{not constant } \forall \psi \in L^1(\Omega) \text{ s.t. } \psi \geq 0\,, \text{ a.e. in } \Omega\right)$$

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Thank you for your attention.