On the effect of spatial heterogeneity in logistic type elliptic equations with nonlinear boundary conditions

Kenichiro Umezu

Ibaraki University

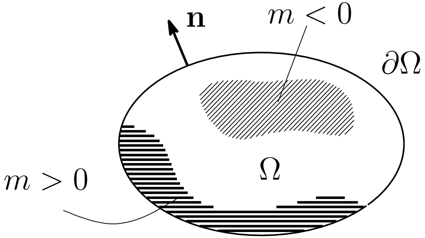
July 4, 2012 in Orlando

Problems

Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^r & \textbf{on } \partial\Omega. \end{cases}$$

Here $\lambda \geq 0$, $p, r > 1$, $m \in C^\theta(\overline{\Omega})$, and $m > 0$ somewhere in Ω.



Problems

Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^r & \text{on } \partial\Omega. \end{cases}$$

Here $\lambda \geq 0$, $p, r > 1$, $m \in C^\theta(\overline{\Omega})$, and $m > 0$ somewhere in Ω.

This is the steady state problem of

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (d\nabla u) + m(x)u - u^p & \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega, \\ (d\nabla u) \cdot \mathbf{n} = u^r & \text{on } (0,\infty) \times \partial\Omega, \end{cases}$$

where $\lambda = 1/d$.

Homogeneous case $p = r$

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^r & \textbf{on } \partial\Omega. \end{cases}$$

The combined nonlinearity appears:

$$\begin{aligned} m(x)u - u^p &\quad \textbf{in } \Omega \quad \textbf{(absorption effect)} \\ u^r &\quad \textbf{on } \partial\Omega \quad \textbf{(blowing-up effect)} \end{aligned}$$

Homogeneous case $p = r$

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^r & \textbf{on } \partial\Omega. \end{cases}$$

In this talk, we restrict our consideration on the homogeneous case

$$1 < p = r < \frac{N}{N-2}.$$

Homogeneous case $p = r$

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^r & \textbf{on } \partial\Omega. \end{cases}$$

In this talk, we restrict our consideration on the homogeneous case

$$1 < p = r < \frac{N}{N-2}.$$

In this case, equivalently for $\lambda > 0$ we consider the scaled problem

$$\begin{cases} -\Delta v = \lambda m(x)v - v^p & \textbf{in } \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = v^p & \textbf{on } \partial\Omega \end{cases}$$

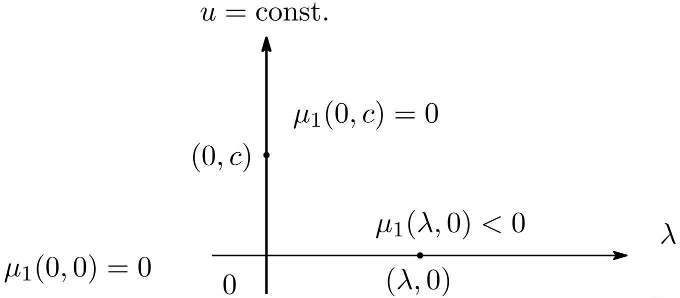
by $v = \lambda^{1/(p-1)}u$**.**

Favorable region case $\int_\Omega m dx \geq 0$

Consider

$$\begin{cases} -\Delta\phi = \lambda(m(x)\phi - pu^{p-1}\phi) + \mu(\lambda, u)\phi & \textbf{in } \Omega, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} = p\lambda u^{p-1}\phi & \textbf{on } \partial\Omega. \end{cases}$$

First we study the favorable case $\int_\Omega m dx \geq 0$, and the zero solution $u = 0$ is unstable for all $\lambda > 0$ (Brown-Lin (1980)).



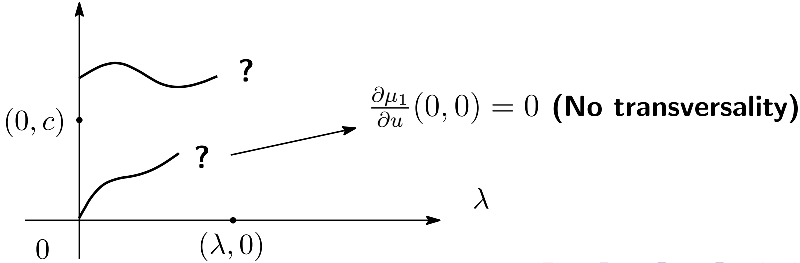
Favorable region case $\int_\Omega m dx \geq 0$

Consider

$$\begin{cases} -\Delta\phi = \lambda(m(x)\phi - pu^{p-1}\phi) + \mu(\lambda, u)\phi & \textbf{in } \Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{n}} = p\lambda u^{p-1}\phi & \textbf{on } \partial\Omega. \end{cases}$$

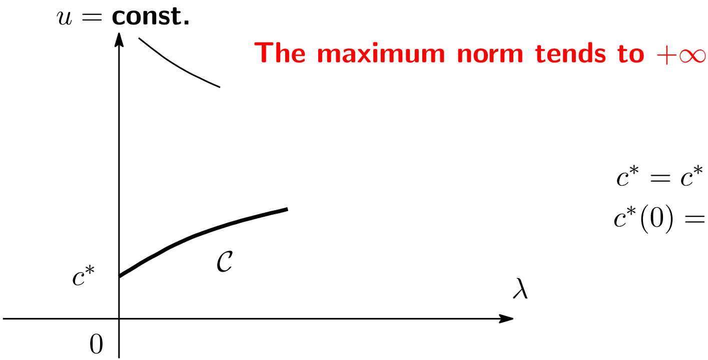
First we study the favorable case $\int_\Omega m dx \geq 0$, and the zero solution $u = 0$ is unstable for all $\lambda > 0$ (Brown-Lin (1980)).

$u = \text{const.}$



Local bifurcation analysis

The local bifurcation analysis was done by (U. (2004)), where it was proved that there exist at least two positive solutions for $\lambda > 0$ small if $\boxed{|\Omega| > |\partial\Omega|}$.



$c^* = c^*(\int_\Omega m dx)$
$c^*(0) = 0$

Local bifurcation analysis

The local bifurcation analysis was done by (U. (2004)), where it was proved that there exist at least two positive solutions for $\lambda > 0$ **small if** $\boxed{|\Omega| > |\partial\Omega|}$**.**

Nonexistence of positive solutions for any $\lambda > 0$ **small enough was proved in the case that** $\boxed{|\Omega| < |\partial\Omega|}$ **(U. (2005)).**

Global analysis for constant coefficients

García-Melián, Morales-Rodrigo, Rossi, and Suáres (2008) studied the case that $\underline{m(x) = m_0 \text{ is a positive constant}}$

$$\begin{cases} -\Delta v = \lambda m_0 v - v^p & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = v^r & \text{on } \partial\Omega, \end{cases}$$

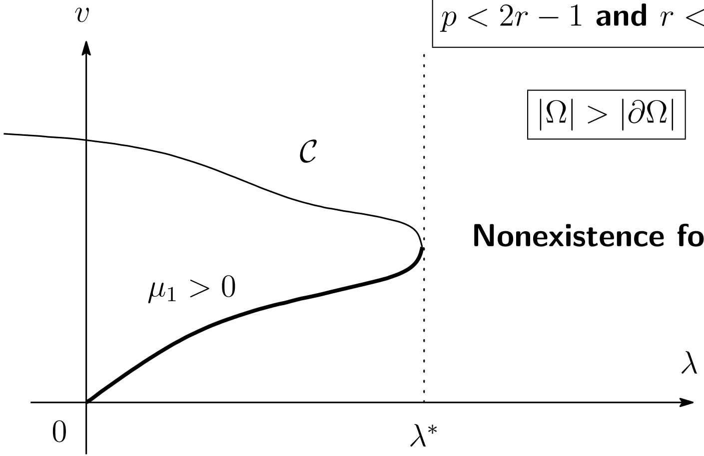
and gave global nature of the bifurcation component of positive solutions at $(\lambda, v) = (0, 0)$. as described in the following diagram.

Meanwhile, they showed that if $p = r$ and $\boxed{|\Omega| \leq |\partial\Omega|}$, then there is no positive solutions for all $\lambda \geq 0$.

Global analysis for constant coefficients

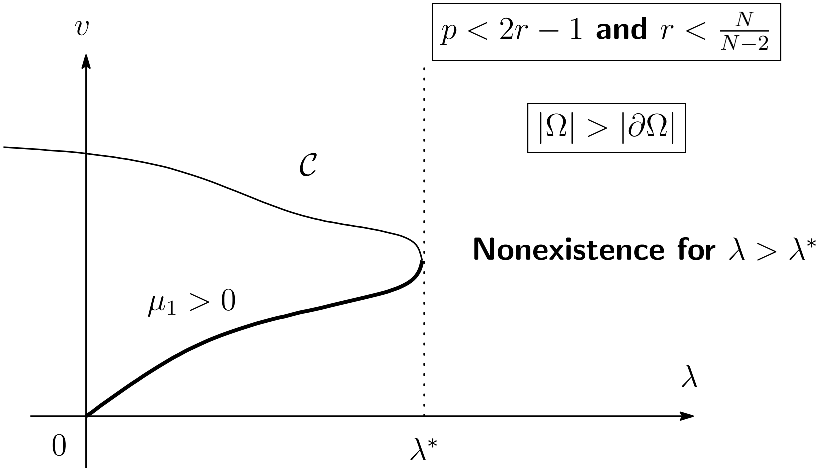
$p < 2r - 1$ **and** $r < \frac{N}{N-2}$

$|\Omega| > |\partial\Omega|$



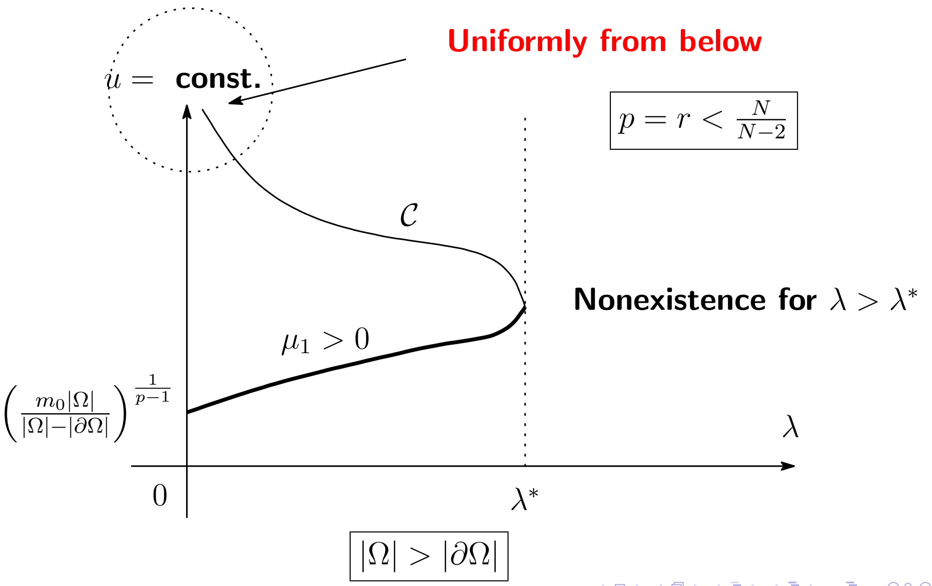
Nonexistence for $\lambda > \lambda^*$

Global analysis for constant coefficients



Set $p = r$ **and convert this to our problem by** $u = \lambda^{-\frac{1}{p-1}} v$**, and we obtain the following.**

Global analysis for constant coefficients



Global analysis for constant coefficients

For the problem

$$\begin{cases} -\Delta u = \lambda(m_0 u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^p & \textbf{on } \partial\Omega, \end{cases}$$

we remark that
(i) the above result can be extended to the case $m > 0$ in $\overline{\Omega}$,

Global analysis for constant coefficients

For the problem

$$\begin{cases} -\Delta u = \lambda(m_0 u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^p & \textbf{on } \partial\Omega, \end{cases}$$

we remark that
(i) the above result can be extended to the case $m > 0$ in $\overline{\Omega}$,

(ii) for the case $m_0 = 0$, there exists at least one positive solution for all $\lambda > 0$ (Chipot, Fila, and Quittner (1991)).

Global analysis for constant coefficients

For the problem

$$\begin{cases} -\Delta u = \lambda(m_0 u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^p & \textbf{on } \partial\Omega, \end{cases}$$

we remark that
(i) the above result can be extended to the case $m > 0$ in $\overline{\Omega}$,

(ii) for the case $m_0 = 0$, there exists at least one positive solution for all $\lambda > 0$ (Chipot, Fila, and Quittner (1991)).

Can we give sufficient conditions of m for the bifurcation component to have turning points ?

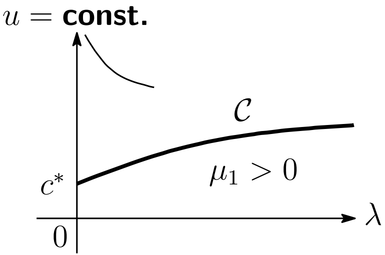
Main results (Existence)

Theorem (No turning points)

Let $p = r < \frac{N}{N-2}$. *Assume* $\int_\Omega m dx \geq 0$ *and* $|\Omega| > |\partial\Omega|$. *Then, there exists a minimal positive solution for all* $\lambda > 0$, *which is asymptotically stable and parametrized continuously by* λ, *provided that*

$$m \leq 0 \quad \textit{on} \quad \partial\Omega.$$

$$c^* = \left(\frac{\int_\Omega m dx}{|\Omega| - |\partial\Omega|} \right)^{\frac{1}{p-1}}$$



Main results (Existence)

Theorem (No turning points)

Let $p = r < \frac{N}{N-2}$. *Assume* $\int_\Omega m dx \geq 0$ *and* $|\Omega| > |\partial\Omega|$. *Then, there exists a minimal positive solution for all* $\lambda > 0$, *which is asymptotically stable and parametrized continuously by* λ, *provided that*

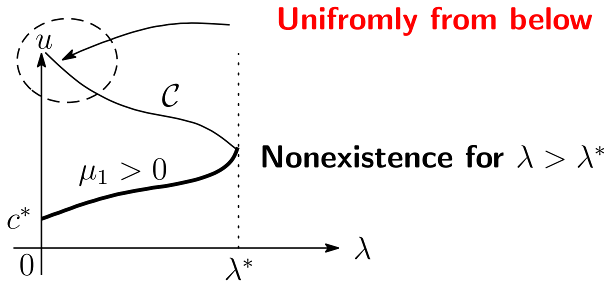
$$m \leq 0 \quad \text{on} \quad \partial\Omega.$$

Here we use the implicit function theorem. We verify the assertion that $\mu_1(\lambda, u) \neq 0$ **for all positive solutions** (λ, u)**.**

Main results (Existence)

Theorem (Turning points exist)

Let $p = r < \frac{N}{N-2}$. Assume $m \geq 0$ in $\overline{\Omega}$ and $|\Omega| > |\partial\Omega|$. Then, there exist at least two positive solutions for all $0 < \lambda < \lambda^*$, one positive solution for $\lambda = \lambda^*$, and no positive solution for any $\lambda > \lambda^*$, provided that $\underline{m(x_0) > 0 \text{ for some } x_0 \in \partial\Omega}$. This means that the bifurcation component from $(0, c^*)$ has a turning point.



Main results (Existence)

Theorem (Turning points exist)

Let $p = r < \frac{N}{N-2}$. *Assume* $m \geq 0$ *in* $\overline{\Omega}$ *and* $|\Omega| > |\partial\Omega|$. *Then, there exist at least two positive solutions for all* $0 < \lambda < \lambda^*$, *one positive solution for* $\lambda = \lambda^*$, *and no positive solution for any* $\lambda > \lambda^*$, *provided that* $m(x_0) > 0$ *for some* $x_0 \in \partial\Omega$. *This means that the bifurcation component from* $(0, c^*)$ *has a turning point.*

In the case $m \geq 0$**,**

$m = 0$ **on** $\partial\Omega$	$\Longrightarrow$	**globally extended in** λ
$m > 0$ **somewhere on** $\partial\Omega$	$\Longrightarrow$	**turning points exist**

Main results (Nonexistence)

Theorem (Nonexitence)

Let $p = r < \frac{N}{N-2}$*. Assume that* $\int_\Omega m dx \geq 0$ *and* $|\Omega| < |\partial\Omega|$ *(possibly* $|\Omega| = |\partial\Omega|$ *when* $\int_\Omega m dx > 0$*). Then,*

(a) there is no positive solution for any $\lambda > 0$*, provided that* $m \leq 0$ *on* $\partial\Omega$*.*

(b) Additionally if $\int_\Omega m dx > 0$*, then for* $\tilde{m}$ *such that* $\tilde{m} > 0$ *somewhere in* Ω*,* $\tilde{m} \leq m$*,* $\int_\Omega \tilde{m} dx > 0$*, and* $\tilde{m} \leq 0$ *on* $\partial\Omega$*, there is no positive solution for any* $\lambda > 0$*.*

Main results (Nonexistence)

Theorem (Nonexitence)

Let $p = r < \frac{N}{N-2}$*. Assume that* $\int_\Omega m dx \geq 0$ *and* $|\Omega| < |\partial\Omega|$ *(possibly* $|\Omega| = |\partial\Omega|$ *when* $\int_\Omega m dx > 0$*). Then,*

(a) there is no positive solution for any $\lambda > 0$*, provided that* $m \leq 0$ *on* $\partial\Omega$*.*

(b) Additionally if $\int_\Omega m dx > 0$*, then for* $\tilde{m}$ *such that* $\tilde{m} > 0$ *somewhere in* Ω*,* $\tilde{m} \leq m$*,* $\int_\Omega \tilde{m} dx > 0$*, and* $\tilde{m} \leq 0$ *on* $\partial\Omega$*, there is no positive solution for any* $\lambda > 0$*.*

This is applicable for the case $m \geq 0$ **and** $\int_\Omega m dx > 0$**, and then there is no positive solutions for any** $\lambda > 0$ **when** $|\Omega| \leq |\partial\Omega|$**.**

Bifurcation direction, super and subcritical

We turn to the case $\int_\Omega m dx < 0$.

Theorem (use of Crandall and Rabinowitz (1971))

Let $p = r$. Assume $\int_\Omega m dx < 0$. Then, positive solutions bifurcate at $(\lambda_1, 0)$ to the left (subcritically) and right (supercritically) respectively if

$$\int_{\partial\Omega} \phi_1^{p+1} ds > \int_\Omega \phi_1^{p+1} dx \text{ and } \int_{\partial\Omega} \phi_1^{p+1} ds < \int_\Omega \phi_1^{p+1} dx,$$

where $\lambda_1 > 0$ is the positive principal eigenvalue of the linearized eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda m(x)\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Main results(Global bifurcation structure)

For the *subcritical* case we have the following.

Theorem (Global bifurcation)

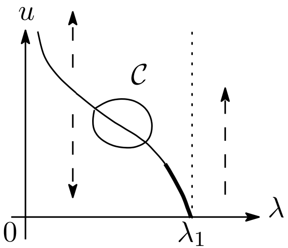
Let $p = r < \frac{N}{N-2}$*. Assume* $\int_\Omega m dx < 0$ *and the bifurcation* $\mathcal{C}$ *at* $(\lambda_1, 0)$ *is subcritical. Then,* $\mathcal{C}$ *is unbounded in* $\mathbb{R} \times C(\overline{\Omega})$ *and the following assertions hold true:*

(a) If we set $J := \{\lambda > 0 : (\lambda, u) \in \mathcal{C}\}$*, then* $J = (0, \lambda_1)$*. Bifurcation from infinity is possible only at* $\lambda = 0$*.*

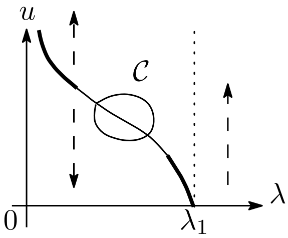
(b) The positive solutions for $0 < \lambda < \lambda_1$ *are all unstable.*

(c) There is no positive solution for $\lambda = \lambda_1$*. Moreover, if* $m(x) \leq 0$ *on* $\partial\Omega$*, then there is no positive solutions for any* $\lambda > \lambda_1$*.*

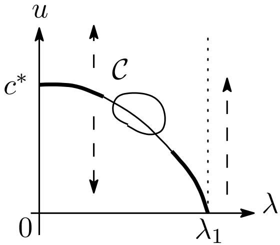
Main results(Global bifurcation structure)



(1) $|\Omega| > |\partial\Omega|$



(2) $|\Omega| = |\partial\Omega|$**,**
$N = 2, 3$**, and** $p = 2$



(3) $|\Omega| < |\partial\Omega|$

Proof for the case $\int_\Omega m dx \geq 0$

- *A priori* **lower positive bound for the positive solutions u of the original problem as $\lambda \to \infty$ (cf. Cantrell and Cosner (1989), (2003) for the singular perturbation problem in the interior of Ω)**
  - **Straighten the boundary and extend the problem by reflection (Lin, Ni, and Takagi (1988))**
  - **Super and subsolutions of uniformly strongly elliptic b.v.p. with the Dirichlet boundary condition (Amann and López-Gómez (1998))**

Proof for the case $\int_\Omega m dx \geq 0$

- *A priori* **lower positive bound for the positive solutions** u **of the original problem as** $\lambda \to \infty$ **(cf. Cantrell and Cosner (1989), (2003) for the singular perturbation problem in the interior of** Ω**)**
  - **Straighten the boundary and extend the problem by reflection (Lin, Ni, and Takagi (1988))**
  - **Super and subsolutions of uniformly strongly elliptic b.v.p. with the Dirichlet boundary condition (Amann and López-Gómez (1998))**
- **Localization of blow up on the boundary (Arrieta and Rodríguez-Bernal(2004))**

Proof for the case $\int_\Omega m dx \geq 0$

- *A priori* **lower positive bound for the positive solutions** u **of the original problem as** $\lambda \to \infty$ **(cf. Cantrell and Cosner (1989), (2003) for the singular perturbation problem in the interior of** Ω**)**
  - **Straighten the boundary and extend the problem by reflection (Lin, Ni, and Takagi (1988))**
  - **Super and subsolutions of uniformly strongly elliptic b.v.p. with the Dirichlet boundary condition (Amann and López-Gómez (1998))**
- **Localization of blow up on the boundary (Arrieta and Rodríguez-Bernal(2004))**
- **Unilateral global bifurcation theory (López-Gómez (2001))**
- *A priori* **upper bounds for positive solutions (Morales-Rodrigo and Suárez (2005))**

Proof for the case $\int_\Omega m dx < 0$

- *A priori* **upper bounds for positive solutions**
- **Local bifurcation analysis at** $(\lambda, v) = (0, 0)$ **for the scaled problem**

$$\begin{cases} -\Delta v = \lambda m(x) v - v^p & \textbf{in } \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = v^p & \textbf{on } \partial\Omega \end{cases}$$

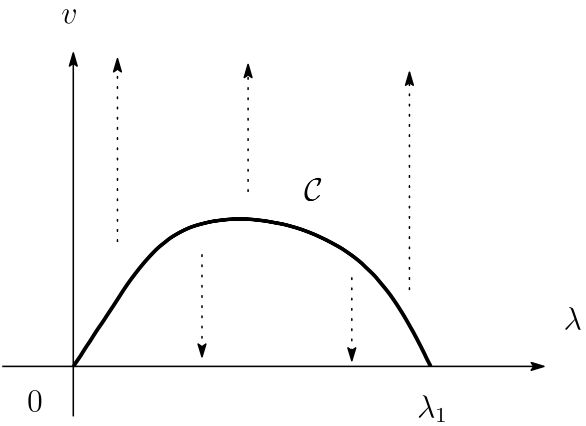
Bounded components

As a corollary, the problem

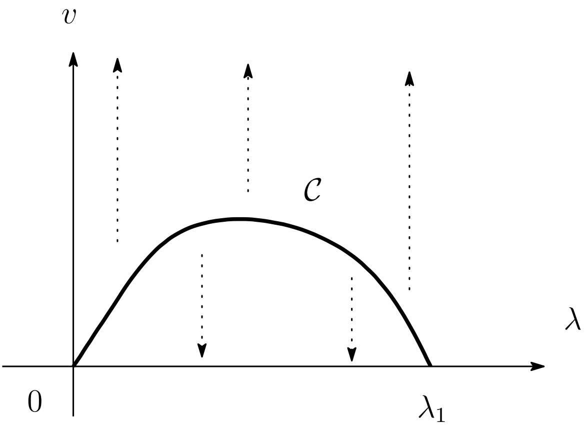
$$\begin{cases} -\Delta v = \lambda m(x) v - v^p & \textbf{in } \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = b v^p & \textbf{on } \partial\Omega \end{cases}$$

with $b > 0$ **has a bounded bifurcation component for** b **large.**

Bounded components



Bounded components



As $\int_\Omega m dx \nearrow 0$, $\mathcal{C}$ must shrink, and finally vanishes.

K.Umezu, J. Differential Equations, 252, (2012), 1146–1168.

Thank you for your attention.