A loop type component of positive solutions of an indefinite concave-convex problem with the Neumann boundary condition

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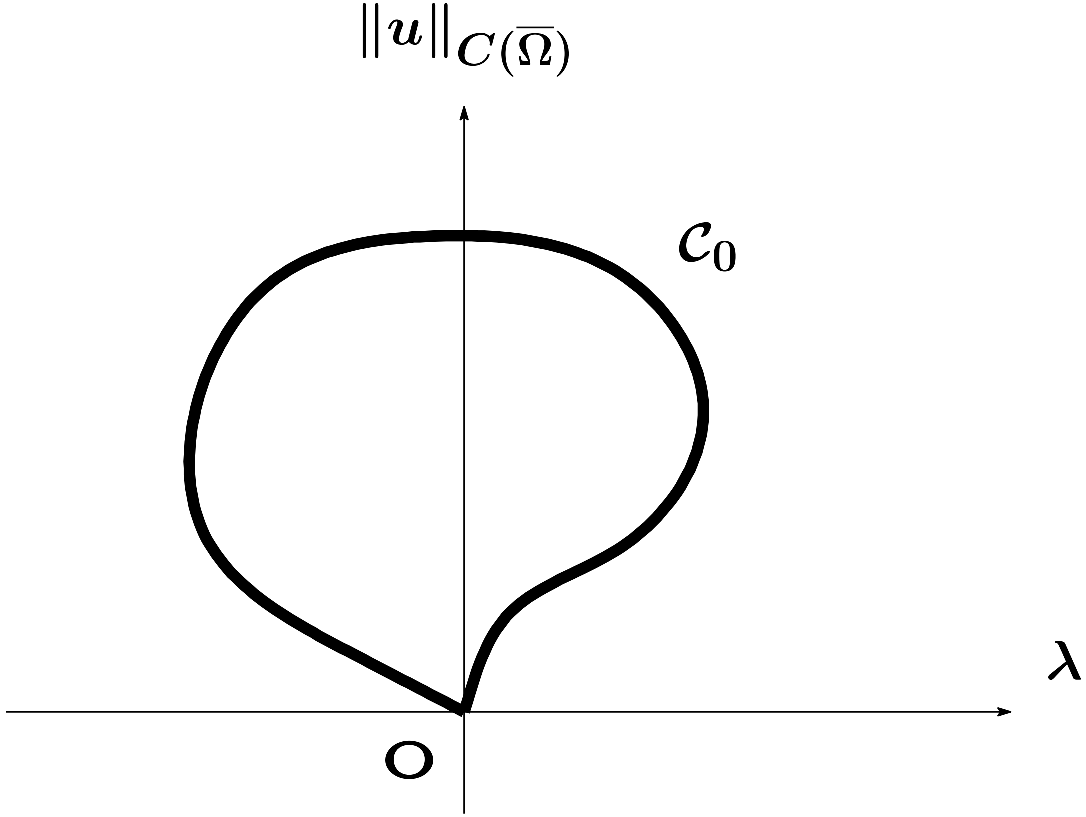
Our problem and purpose. Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a smooth bounded domain. Consider

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda b(x) u^{q-1} + a(x) u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here,

- $\lambda \in \mathbb{R}$ is a bifurcation parameter;
- $1 < q < 2 < p$;
- $a, \quad b \in C^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$, change sign.

Our purpose: to discuss the existence of a loop type component of nontrivial non-negative solutions for (P_λ) under some certain conditions.


$\|u\|_{C(\overline{\Omega})}$
$\mathcal{C}_0$
λ
O

- **Ambrosetti, Brezis, and Cerami (1994)**

  $a, b \equiv 1$; **Dirichlet; Existence, nonexistence, and multiplicity; Sub and supersolutions; Variational technique.**

- **Delgado and Suárez (2003)**

  Non self-adjoint, uniformly elliptic operators; $b \equiv 1$; **Dirichlet; Unbounded component of non-negative solutions; bifurcation; Leray-Schauder degree.**

- **de Figueiredo, Gossez, and Ubilla (2006)**

  A wide class of concave-convex type with $b \geq 0$; **Dirichlet.**

- **Korman (2013)**

  Ω **is a ball or an annulus; Dirichlet;** $a, b \equiv 1$; **Solution curve.**

(continued)

$$\int_\Omega \lambda b u^{q-1} + a u^{p-1} = 0 \implies b \not\geq 0 \quad \text{or} \quad a \not\geq 0 \quad \text{if} \quad u > 0.$$

- **Tarfulea (1998)**

  $b \equiv 1$; **Neumann**; $\int_\Omega a < 0$ is necessary and sufficient for a positive solution when $\lambda > 0$; Sub and supersolutions.

- **Alama (1999)**

  $a \equiv 1$; **Neumann**; Existence, nonexistence, and multiplicity; Dead core issue (when b changes sign).

Our interest: the case when a, b change sign. We may assume that

$$\int_\Omega b \leq 0. \qquad (\text{because} \quad \lambda b(x) = (-\lambda)(-b(x)))$$

Our argument proceeds with:

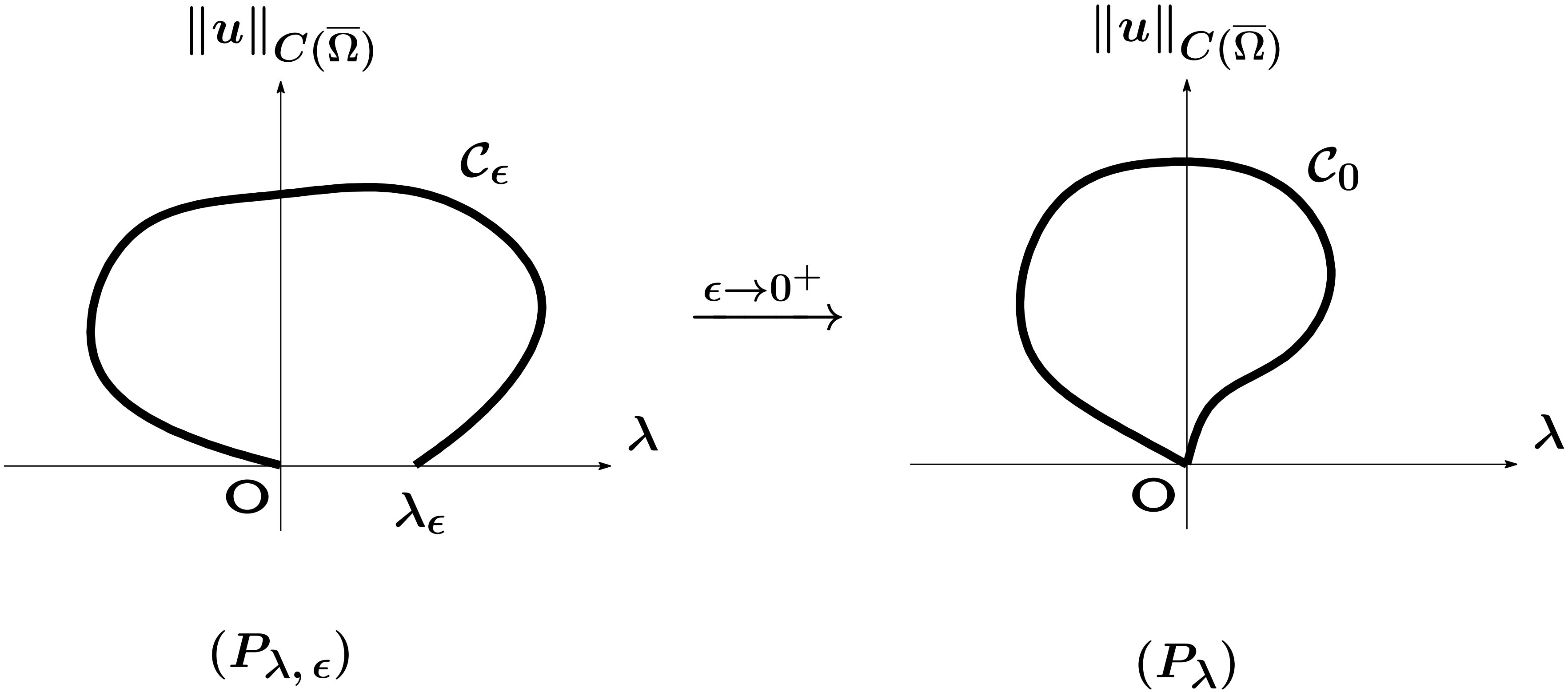
$\surd$ a regularized problem for (P_λ) at $u = 0$ with $\epsilon > 0$:

$$(P_{\lambda,\epsilon}) \quad \begin{cases} -\Delta u = \lambda(b(x) - \epsilon)(u + \epsilon)^{q-2}u + a(x)u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

$\surd$ a priori bounds of the norm $\|\cdot\|_{C(\overline{\Omega})}$ for non-negative solutions, following an argument proposed by Amann and López-Gómez,

$\surd$ a priori bounds of parameter λ for nontrivial non-negative solutions,

$\surd$ a topological analysis introduced by Whyburn


$\|u\|_{C(\overline{\Omega})}$
$\mathcal{C}_\epsilon$
λ
O
λ_ϵ
$\epsilon \to 0^+$
$\|u\|_{C(\overline{\Omega})}$
$\mathcal{C}_0$
λ
O
$(P_{\lambda,\epsilon})$
(P_λ)

Our assumptions. Set

$$\Omega_{\pm}^{a} = \{x \in \Omega : a \gtrless 0\}, \quad \Omega_{\pm}^{b} = \{x \in \Omega : b \gtrless 0\},$$

and then assume

(H_0) $\Omega_{+}^{a} \cap \Omega_{+}^{b} \neq \emptyset, \quad \Omega_{+}^{a} \cap \Omega_{-}^{b} \neq \emptyset$;

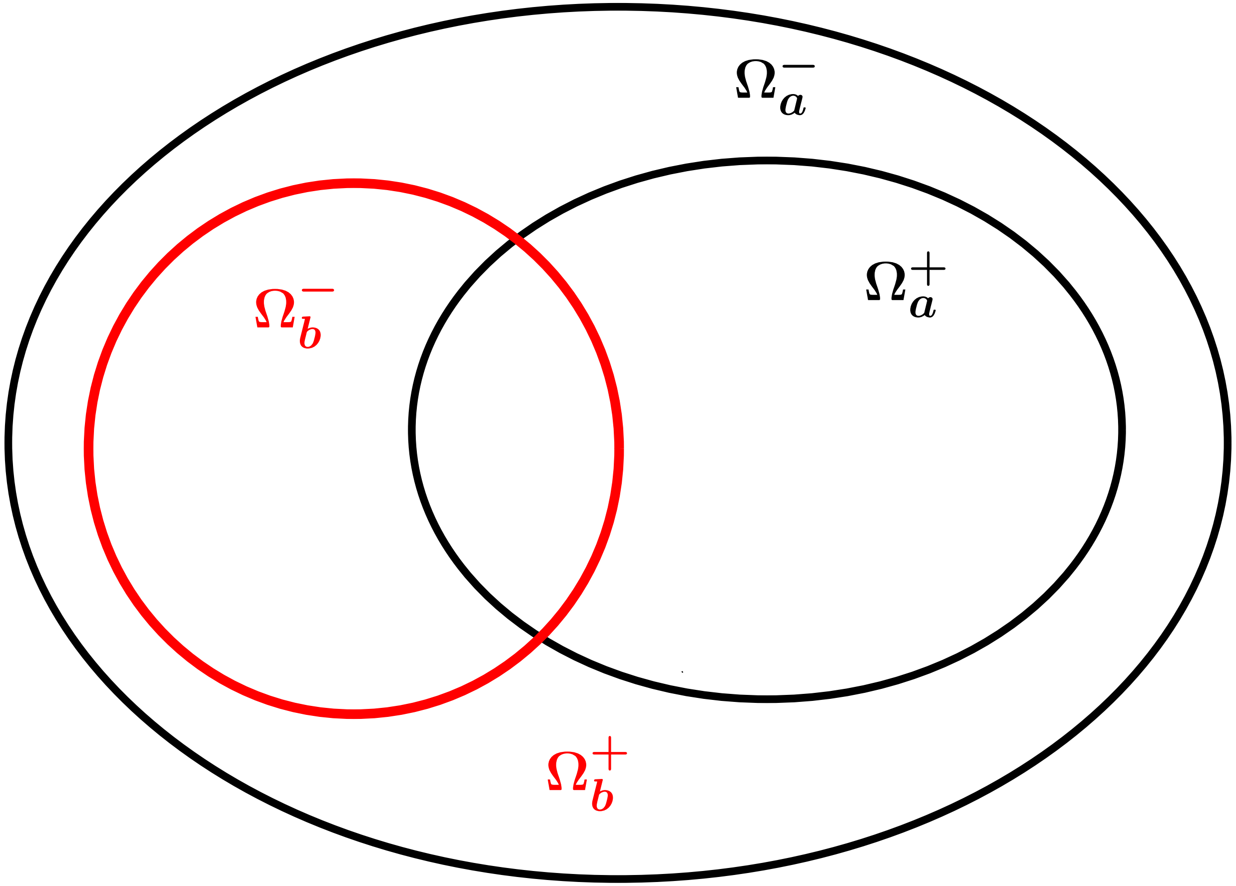
(H_1) $\Omega_{\pm}^{a}$ are subdomains of Ω with smooth boundaries, and satisfy $\overline{\Omega_{+}^{a}} \subset \Omega$ and $\overline{\Omega_{+}^{a}} \cup \Omega_{-}^{a} = \Omega$;

(H_2) Under (H_1), there exists α^{+}, continuous, positive, and bounded away from zero in a tubular neighborhood of $\partial\Omega_{+}^{a}$ and $\gamma > 0$ such that

$$a^{+}(x) = \alpha^{+}(x)\, \mathrm{dist}(x, \partial\Omega_{+}^{a})^{\gamma}, \quad 2 < p < \min\left\{\frac{2N}{N-2}, \frac{2N+\gamma}{N-1}\right\};$$

(H_3) $\Omega_{\pm}^{b}$ are subdomains of Ω.

An example of Ω satisfying (H_0), (H_1), and (H_3):



Theorem. Assume that $\int_\Omega a < 0$. If (H_k), $k = 0, 1, 2, 3$, are satisfied, then (P_λ) admits a loop type, bounded component $\mathcal{C}_0 = \{(\lambda, u)\}$ (closed and connected subset) of nontrivial non-negative solutions such that:

(1) $\mathcal{C}_0$ joins $(0,0)$ to itself;

(2) $\mathcal{C}_0 \neq \{(0,0)\}$;

(3) $\mathcal{C}_0$ does not meet $(\lambda, 0)$ for any $\lambda \neq 0$;

(4) There exists $\delta > 0$ such that $\mathcal{C}_0$ does not contain any positive solution u of (P_λ) with $\lambda = 0$ satisfying $\|u\|_{C(\overline{\Omega})} \leq \delta$.

A regularization argument. We choose $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$, then $\Omega_+^{b-\epsilon} \neq \emptyset$. For such $\epsilon > 0$, we consider

$$(P_{\lambda,\epsilon}) \qquad \begin{cases} -\Delta u = \lambda(b(x) - \epsilon)(u + \epsilon)^{q-2}u + a(x)u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

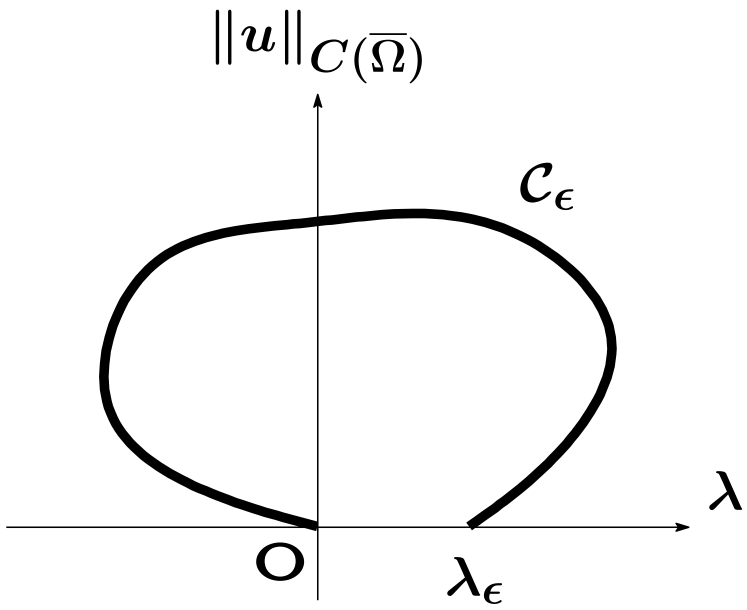
The linearized eigenvalue problem for $(P_{\lambda,\epsilon})$ at $u = 0$

$$\begin{cases} -\Delta \phi = \lambda(b - \epsilon)\epsilon^{q-2}\phi & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \mathrm{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

has exactly two principal eigenvalues $0, \lambda_\epsilon$, where $\lambda_\epsilon > 0$, and $(0,0)$, $(\lambda_\epsilon, 0)$ both satisfy the transversality condition in the local bifurcation theory by Crandall and Rabinowitz. Moreover, we can verify that

$$\lambda_\epsilon \longrightarrow 0 \quad \text{as } \epsilon \to 0^+.$$

- √ The unilateral global bifurcation theorem by López-Gómez can be applied to $(\lambda_\epsilon, 0)$ to obtain a component of positive solutions of $(P_{\lambda,\epsilon})$ bifurcating at $(\lambda_\epsilon, 0)$. Moreover, if the component $\mathcal{C}_\epsilon$ is not unbounded in $\mathbb{R} \times C(\overline{\Omega})$, then it meets $(0, 0)$.
- √ The bifurcation at $(0, 0)$ is to the left (subcritical), since $\int_\Omega a < 0$. Consequently, $\mathcal{C}_\epsilon$ cuts the vertical axis.



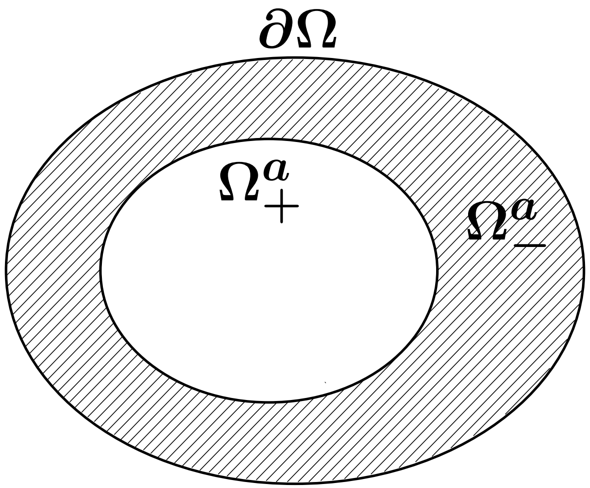
Proposition (A priori bounds). Assume that (H_1) holds. Let $\Lambda > 0$. Suppose

$$\exists\, C_1 > 0 \quad \text{s.t.} \quad \|u\|_{C(\overline{\Omega_+^a})} \le C_1$$

for all non-negative solutions u of $(P_{\lambda,\epsilon})$ with $\lambda \in [0, \Lambda]$ and small $\epsilon > 0$. Then,

$$\exists C_2 > 0 \quad \text{s.t.} \quad \|u\|_{C(\overline{\Omega})} \le C_2$$

for such non-negative solutions.



Proof. Consider the concave problem

$$\begin{cases} -\Delta v = -a^{-}(x)v^{p-1} + \lambda b^{+}(x)(v+\epsilon)^{q-2}v & \text{in } \Omega_{-}^{a}, \\ v = C_1 & \text{on } \partial\Omega_{+}^{a}, \\ \frac{\partial v}{\partial \mathrm{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

If u is a nontrivial non-negative solution of $(P_{\lambda,\epsilon})$ with $\lambda \in [0, \Lambda]$ and small $\epsilon > 0$, then u is a subsolution of this problem. To construct a supersolution, we consider the unique positive solution w_0 of

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega_{-}^{a}, \\ w = 0 & \text{on } \partial\Omega_{+}^{a}, \\ \frac{\partial w}{\partial \mathrm{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

If we set $w = C(w_0 + 1)$, $C > 0$, then w is a supersolution of the concave problem, provided C is large. Here C does not depend on λ, ϵ. The comparison principle shows $u \le w$ in $\overline{\Omega_{-}^{a}}$, as desired.

Proposition (A priori bounds 2). Assume that $\Omega_+^a \cap \Omega_+^b \neq \emptyset$. Then, there exist $\overline{\lambda} > 0$ and $\epsilon_0 > 0$ such that $(P_{\lambda,\epsilon})$ has no nontrivial non-negative solutions for any $\lambda \geq \overline{\lambda}$ and $\epsilon \in (0, \epsilon_0]$.

Remark. Additionally assume that $\Omega_+^a \cap \Omega_-^b \neq \emptyset$. Then, this proposition can be trivially extended to the case $|\lambda| \geq \overline{\lambda}$, since we note that

$$-\Delta u = a(x)u^{p-1} + (-\lambda)\{-(b(x) - \epsilon)\}(u + \epsilon)^{q-2}u.$$

A sketch of proof. Choose a ball B such that $\overline{B} \subset \Omega$, satisfying that

$$a(x), \quad b(x) - \epsilon_0 > 0, \quad x \in \overline{B}.$$

Consider an eigenfunction $\phi > 0$ associated with the first eigenvalue $\lambda_1 > 0$ of

$$-\Delta\phi = \lambda a(x)\phi \;\text{ in } B, \quad \phi|_{\partial B} = 0.$$

Let $\epsilon \in (0, \epsilon_0]$. Then, the divergence theorem shows that

$$\int_B u^{q-1}\phi \left(a(x)u^{p-q} + \lambda(b(x) - \epsilon) \left(\frac{u}{u+\epsilon}\right)^{2-q} - \lambda_1 a(x)u^{2-q} \right) < 0.$$

We observe that

$$\left(\frac{u}{u+\epsilon}\right)^{2-q} \geq c_0\, u^{2-q}, \quad 0 \leq u \leq u_0.$$

We then get a contradiction when $\lambda \to \infty$.

Topological analysis. Let

$$\liminf_{\epsilon} \mathcal{C}_\epsilon = \left\{ (\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : \lim_{\epsilon} \operatorname{dist}\left((\lambda, u), \mathcal{C}_\epsilon\right) = 0 \right\},$$
$$\limsup_{\epsilon} \mathcal{C}_\epsilon = \left\{ (\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : \liminf_{\epsilon} \operatorname{dist}\left((\lambda, u), \mathcal{C}_\epsilon\right) = 0 \right\}.$$

Then, we can show (Whyburn) that

$$(0,0) \in \liminf_{\epsilon} \mathcal{C}_\epsilon \subset \limsup_{\epsilon} \mathcal{C}_\epsilon =: \mathcal{C}_0 : \text{ a component.}$$

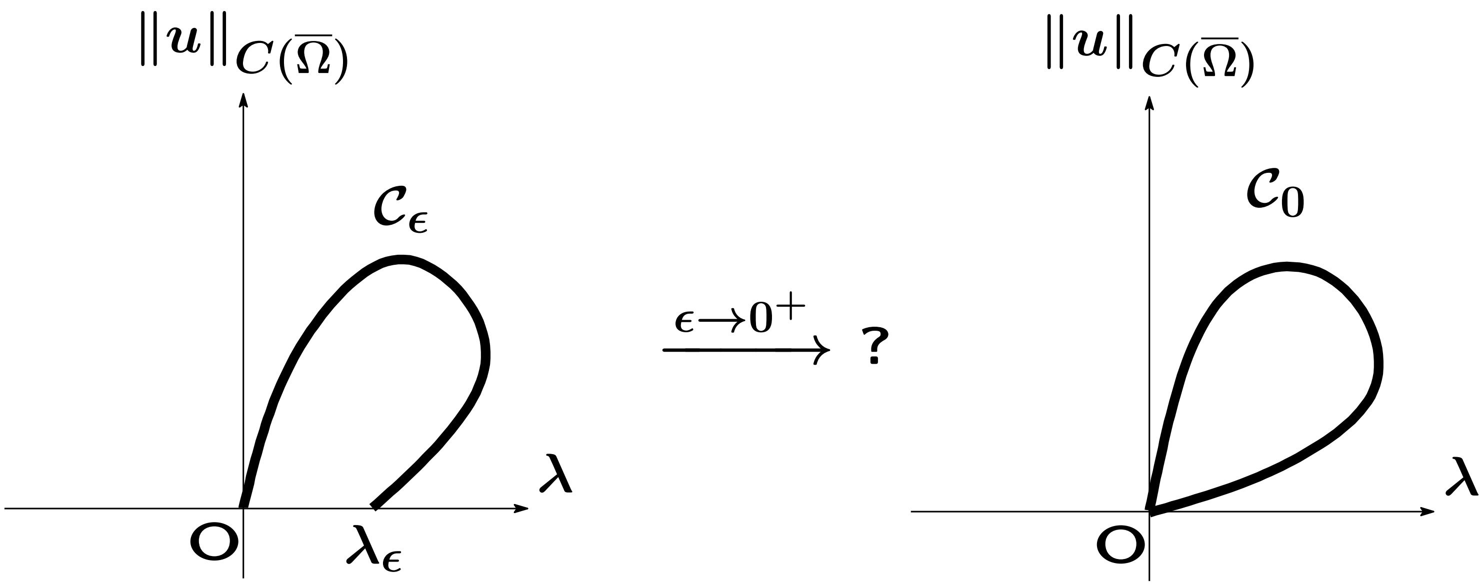
Finally, $\mathcal{C}_0$ is as desired, verifying that

√ $\mathcal{C}_0$ does not meet any $(\lambda, 0)$ with $\lambda \neq 0$.

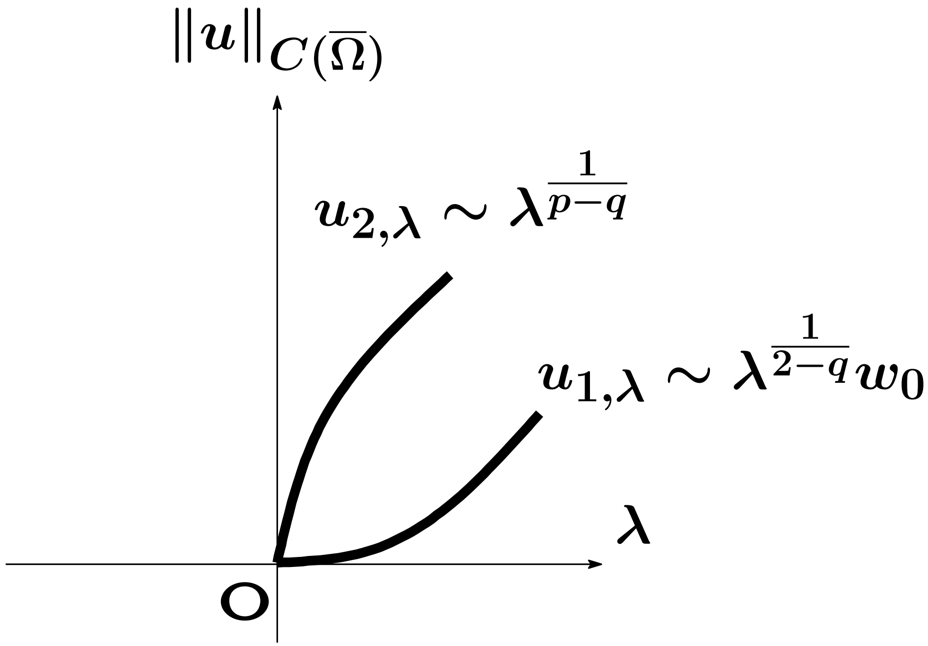
√ (P_λ) with $\lambda = 0$, i.e., $-\Delta u = a(x) u^{p-1}$ in Ω with $\frac{\partial u}{\partial \mathrm{n}} = 0$ on $\partial\Omega$ has no positive solutions small for the case $\int_\Omega a < 0$.

Open problems. When $\int_\Omega a \geq 0$, the same bifurcation argument can be carried out for $(P_{\lambda,\epsilon})$, and moreover, there exist no positive solutions of $(P_{\lambda,\epsilon})$ with $\lambda = 0$:

$$-\Delta w = a(x) w^{p-1} \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \mathrm{n}} = 0 \quad \text{on } \partial\Omega.$$



Let $p < \frac{2N}{N-2}$ if $N > 3$. We can prove that if $\int_\Omega a > 0 > \int_\Omega b$, then (P_λ) has two nontrivial non-negative, variational solutions $u_{1,\lambda},\ u_{2,\lambda}$ for small $\lambda > 0$ such that $u_{1,\lambda} < u_{2,\lambda}$, and both converge to 0.



Here, w_0 is a nontrivial non-negative, least energy solution of

$$-\Delta w = b(x) w^{q-1} \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \mathrm{n}} = 0 \quad \text{on } \partial\Omega.$$

Thank you for your attention.