ロジスティックタイプの非線形楕円型境界値問題に対する正値解の大域的分岐構造について

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Bifurcation and stability

Consider the stationary solutions of the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(\lambda, u) & \textbf{in } (0,\infty)\times\Omega, \\ u(0,x) = u(x) \geq 0 & \textbf{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = g(\lambda, u) & \textbf{on } (0,\infty)\times\partial\Omega \end{cases}$$

with $f(\lambda, 0) = g(\lambda, 0) = 0$**.**

Bifurcation and stability

Study the positive solutions of the problem

$$\begin{cases} -\Delta u = f(\lambda, u) & \textbf{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = g(\lambda, u) & \textbf{on } \partial\Omega. \end{cases}$$

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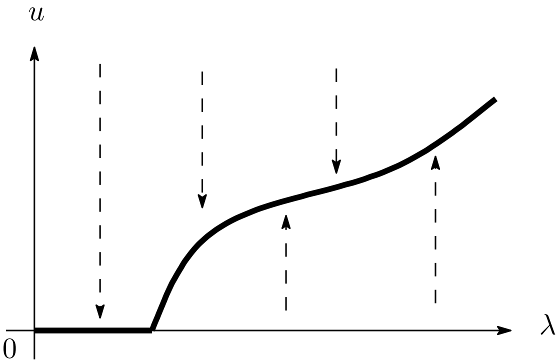

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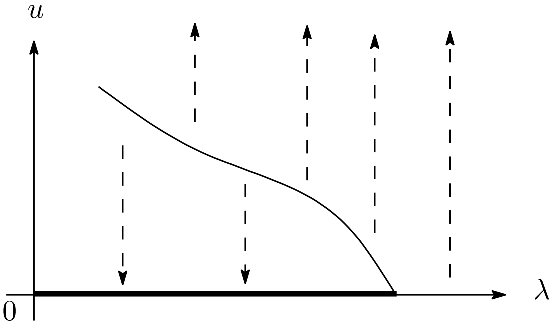

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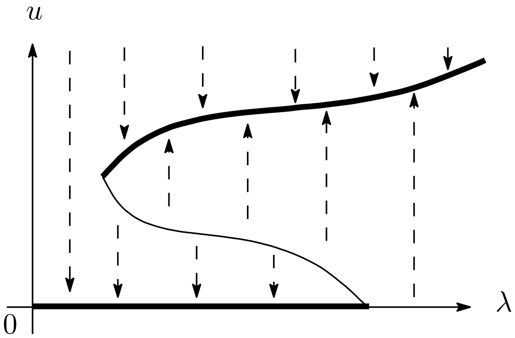

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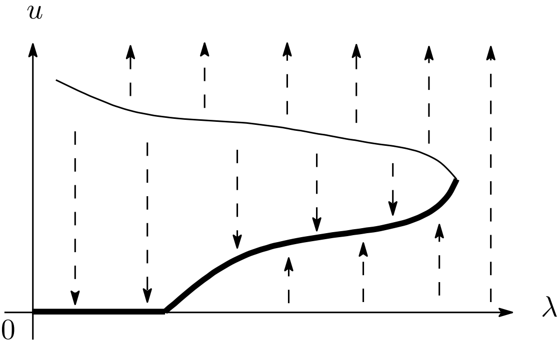

Problem

$\Omega \subset \mathbb{R}^N, N \geq 2$, **滑らかな有界領域．**

$$u(x) \quad \begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \textbf{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda b(x) u^r & \textbf{on } \partial\Omega. \end{cases}$$

$\lambda \geq 0$, $p, r > 1$, $m \in C^\theta(\overline{\Omega})$, $m > 0$ **somewhere in** Ω,
$b \in C^{1+\theta}(\partial\Omega)$, $b > 0$ **on** $\partial\Omega$.

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$$\frac{\partial u}{\partial t} = 0 \implies d = \frac{1}{\lambda}$$

$$u(t,x) \begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (d\nabla u) + m(x)u - u^p & \textbf{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) \geq 0 & \textbf{in } \Omega, \\ (d\nabla u) \cdot \mathbf{n} = b(x)u^r & \textbf{on } (0,\infty) \times \partial\Omega. \end{cases}$$

Homogeneous case $p = r$

2 種の非線形性

$$m(x)u - u^p \quad \textbf{in } \Omega \implies \textbf{吸収型（absorption）}$$
$$b(x)u^r \quad \textbf{on } \partial\Omega \implies \textbf{爆発型（blowing-up）}$$

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設定

$1 < p = r < \frac{N}{N-2}$ **(homogeneous case),** $\quad b \equiv 1.$

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Scaling

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u^p & \text{on } \partial\Omega, \end{cases}$$

$$\overset{v=\lambda^{1/(p-1)}u}{\Longrightarrow} \begin{cases} -\Delta v = \lambda m(x)v - v^p & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = v^p & \text{on } \partial\Omega. \end{cases}$$

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$2 < p + 1 < \frac{2(N-1)}{N-2}$,

$$H^1(\Omega) \subset L^{\frac{2(N-1)}{N-2}}(\partial\Omega), \quad \text{continuous},$$
$$H^1(\Omega) \subset L^{p+1}(\partial\Omega), \quad \text{compact}.$$

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設定

$1 < p = r < \frac{N}{N-2}$ **(homogeneous case),** $\quad b \equiv 1.$

$p = 2$ **(logistic)** $\implies 2 < \frac{N}{N-2}$

$$N = 2,\ 3$$

先行研究 (Ω で吸収型，$\partial\Omega$ で爆発型の研究)

べき乗型 (Ω, $\partial\Omega$)

Chipot, Fila, and Quittner (1991) べき乗型，定数係数

Pflüger (1999) べき乗型，変数係数，coercive and noncoercive cases

$$\begin{cases} (-\Delta + q(x))u = a(x)u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = b(x)u^r & \text{on } \partial\Omega. \end{cases}$$

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$\mu_1(-\Delta + q, N) > 0$ **とする．** $p = r$ **のとき，** $b > 0$ **somewhere ならば** $\exists$ **positive solution.**

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + qu^2) - \frac{1}{p+1}\int_\Omega a|u|^{p+1} - \frac{1}{p+1}\int_{\partial\Omega} b|u|^{p+1}$$

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Berestycki, Capuzzo-Dolcetta, and Nirenberg (1995) coercive and noncoercive cases, variational approach

Ambrosetti, Brezis, and Cerami (1994) べき乗型 (Ω, Ω), bifurcation approach, super and subsolutions, variational approach

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logistic(Ω), べき乗型 ($\partial\Omega$)

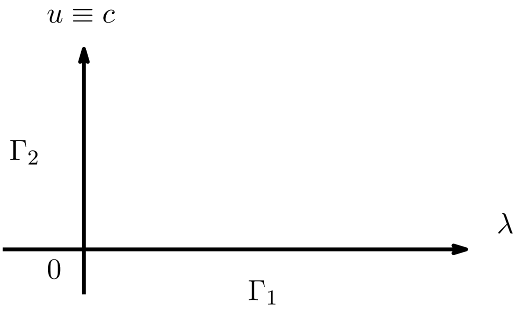
U.(2004) local bifurcation analysis

García-Melián, Morales-Rodrigo, Rossi, and Suárez (2008) 定数係数 $m > 0$, global bifurcation analysis

Two trivial lines of solutions

$$\begin{cases} -\Delta u = \lambda(m(x)u - u^p) & \textbf{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u^p & \textbf{on } \partial\Omega. \end{cases}$$

$$\Gamma_1 = \{(\lambda, u) : u = 0\}, \quad \Gamma_2 = \{(\lambda, u) : \lambda = 0,\ u \textbf{ is a constant}\}.$$



Bifurcation analysis from $(\lambda, 0)$, linearized

$u = 0$ **における線形化固有値問題**

$$\begin{cases} -\Delta\varphi - \lambda m(x)\varphi = \mu(\lambda)\varphi & \textbf{in } \Omega, \\ \frac{\partial\varphi}{\partial\mathbf{n}} = 0 & \textbf{on } \partial\Omega. \end{cases}$$

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The smallest $\mu_1(\lambda)$ **(**$\mu_1(0) = 0$**)**

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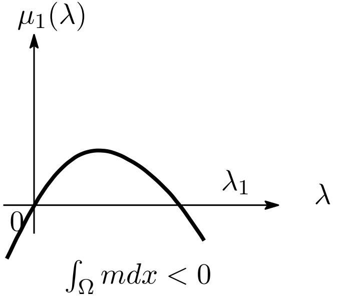
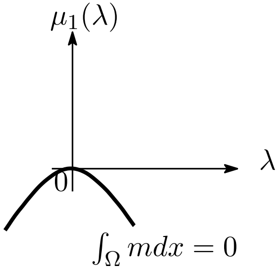
The smallest $\mu_1(\lambda)$ **(**$\mu_1(0) = 0$**)**
Brown and Lin (1980) の結果：

$$\mu_1(\lambda) < 0, \quad \lambda > 0 \qquad \textbf{if} \int_\Omega m dx \geq 0,$$

$$\mu_1(\lambda) \begin{cases} > 0, & 0 < \lambda < \lambda_1 \\ = 0, & \lambda = \lambda_1 \\ < 0, & \lambda_1 < \lambda \end{cases} \qquad \textbf{if} \int_\Omega m dx < 0.$$

$\lambda \mapsto \mu_1(\lambda)$ のグラフ





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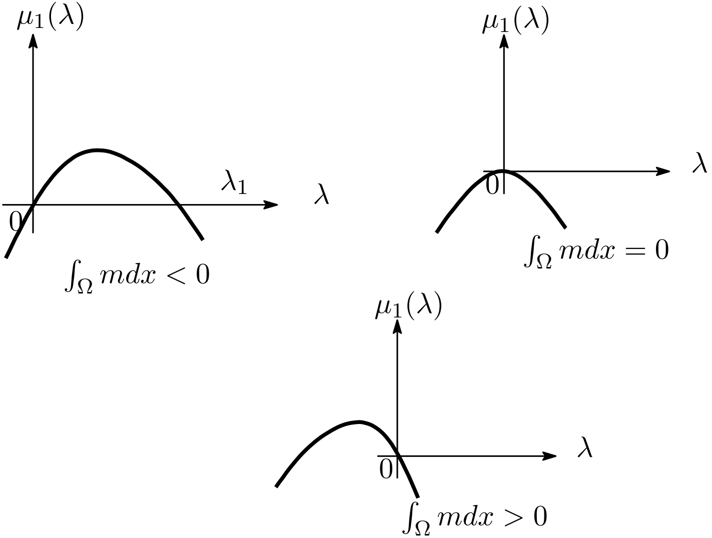

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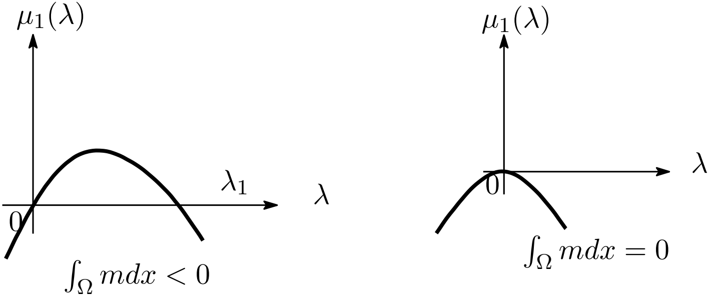

$\int_\Omega m dx < 0$ **and** $0 < \lambda < \lambda_1$ **(coercive)**

$\Longrightarrow$ **at least one positive solution (Pflüger)**

分岐が起こるための必要条件

- **自明な枝** $\Gamma_1 = \{(\lambda, 0) : \lambda \geq 0\}$,
  $\Gamma_2 = \{(0, c) : c \geq 0, \text{constant}\}$ **からの分岐**

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- **3 つの可能性**

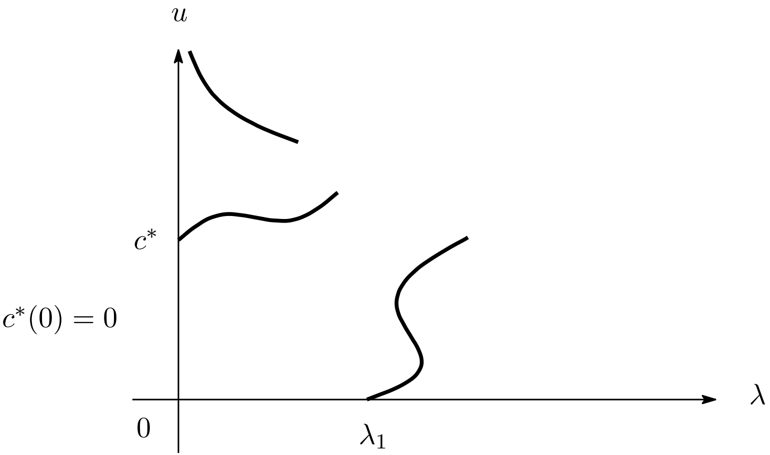
$$\bullet(\lambda_1, 0) \in \Gamma_1,\ \textbf{ただし} \int_\Omega m dx < 0$$

$$\bullet \left(0, \left(\frac{\int_\Omega m dx}{|\Omega| - |\partial\Omega|}\right)^{1/(p-1)}\right) \in \Gamma_2,$$

$$\textbf{特に} \int_\Omega m dx = 0 \ \textbf{のときは}\ (0, 0).$$

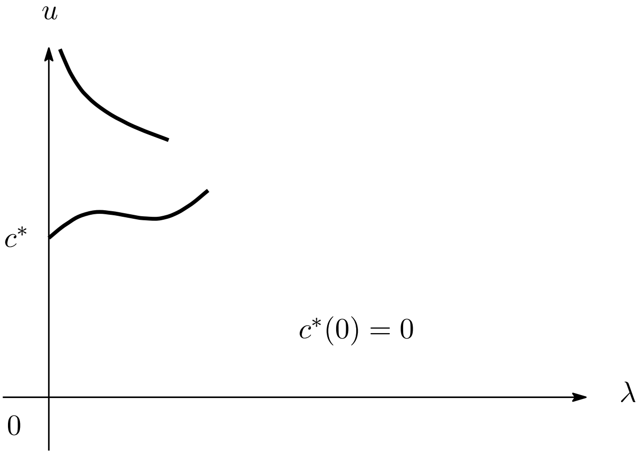
$$\bullet(0, +\infty) \quad \textbf{bifurcation from infinity}$$

$$c^*\left(\int_\Omega m dx\right) := \left(\frac{\int_\Omega m dx}{|\Omega| - |\partial\Omega|}\right)^{1/(p-1)}$$



分岐の 3 つの可能性

Case $\int_\Omega m dx \geq 0$



Case $m(x) \geq 0$ in $\overline{\Omega}$

- $m > 0$, **constant (García-Melián, Morales-Rodrigo, Rossi, and Suáres (2008))**

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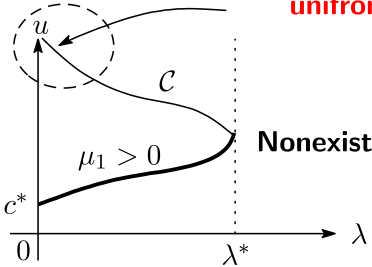
- $|\Omega| > |\partial\Omega|$ **and** $m > 0$ **in** $\overline{\Omega}$

unifromly from below



Nonexistence for $\lambda > \lambda^*$

Remark. For the case $m(x) \equiv 0$ **in** $\overline{\Omega}$**, at least one positive solution for each** $\lambda > 0$ **(CFQ(1991))**

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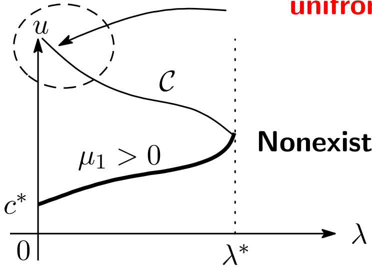


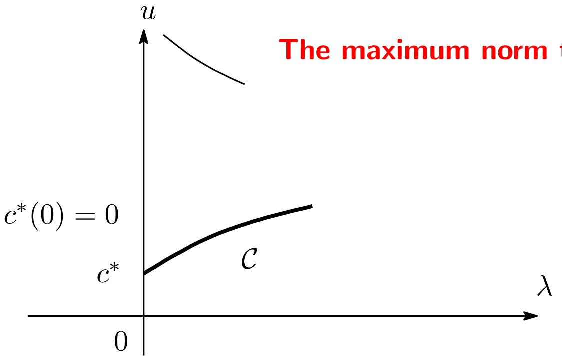
Nonexistence for $\lambda > \lambda^*$

Remark. **For the case $m(x) \equiv 0$ in $\overline{\Omega}$, at least one positive solution for each $\lambda > 0$ (CFQ(1991))**

- $|\Omega| \leq |\partial\Omega| \implies$ **Nonexistence of positive solutions for all $\lambda > 0$**

Case m is sign-changing

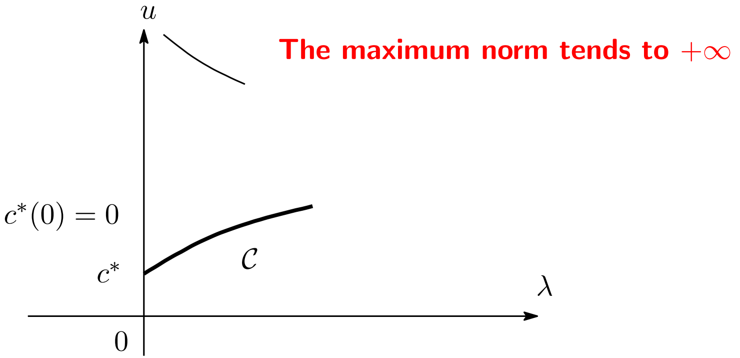
- $|\Omega| > |\partial\Omega| \implies$ **local analysis (U.(2004))**



The maximum norm tends to $+\infty$

Case m is sign-changing

- $|\Omega| > |\partial\Omega| \implies$ **local analysis (U.(2004))**



- $|\Omega| < |\partial\Omega| \implies$ **Nonexistence of positive solutions for any** $\lambda > 0$ **small enough (U.(2005)).**

主結果1 Extended to the whole parameter space

Theorem 1

Assume $\int_\Omega m dx \geq 0$ and $|\Omega| > |\partial\Omega|$. If $m(x) \leq 0$ on $\partial\Omega$, then there exists at least one positive solution for any $\lambda > 0$.

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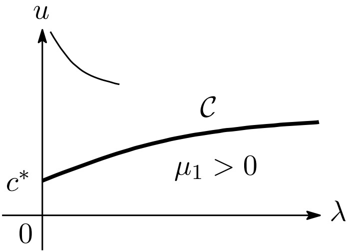

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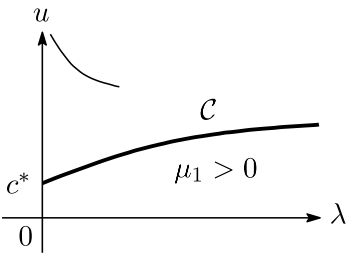


Remark 1. **Consider the case $m \geq 0$ in $\overline{\Omega}$ and $m = 0$ on $\partial\Omega$. Then, Theorem 1 applies.**

主結果2 Existence of a turning back point

Theorem 2

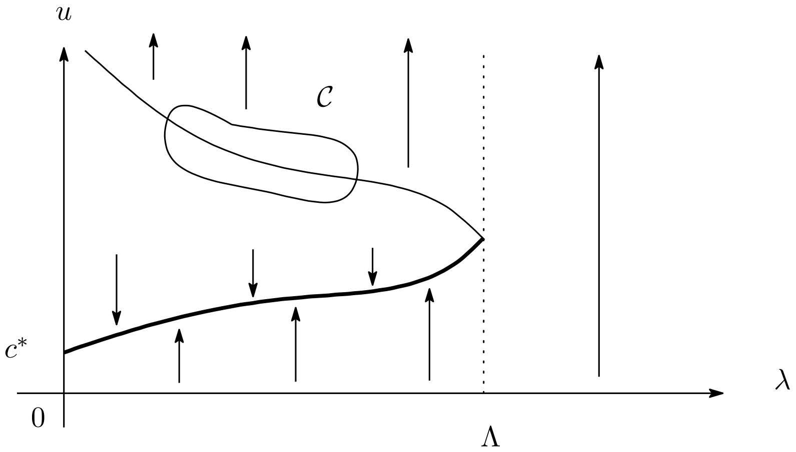
Suppose that $m \geq 0$ in $\overline{\Omega}$. Let Ω_δ, $\delta > 0$, be a tubular neighborhood of $\partial\Omega$, and let m satisfy the condition (M_δ)

$$\exists \delta > 0 \ \ \textbf{s.t.} \ \ m > 0 \ \ \textbf{in } \Omega_\delta.$$

Then we have

$$\Lambda := \sup\{\lambda > 0 : \exists \textbf{ positive solution for } \lambda\} < \infty$$

and moreover, there exist at least two positive solutions for $0 < \lambda < \Lambda$ and one positive solution for $\lambda = \Lambda$.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主結果3 Nonexistence

Theorem 3

Assume that $\int_\Omega m dx \geq 0$ and $|\Omega| < |\partial\Omega|$ (possibly $|\Omega| = |\partial\Omega|$ when $\int_\Omega m dx > 0$). Then we have the following two assertions.

(a) If $m \leq 0$ on $\partial\Omega$, then there is no positive solution for any $\lambda > 0$.

(b) When $\int_\Omega m dx > 0$, if we can choose $\tilde{m}$ such that $\tilde{m} > 0$ somewhere in Ω, $\tilde{m} \leq m$, $\int_\Omega \tilde{m} dx > 0$, and $\tilde{m} \leq 0$ on $\partial\Omega$, then there is no positive solution for any $\lambda > 0$.

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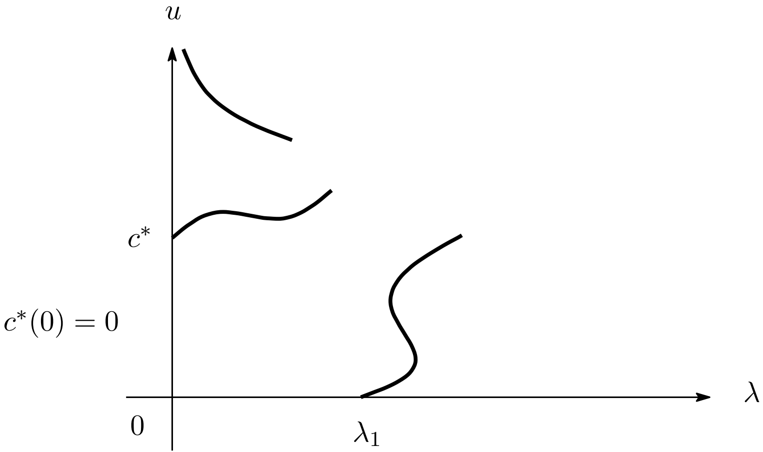
Remark. **Consider the case $m \geq 0$. Then, Theorem 3(b) applies.**

Case $\int_\Omega m dx < 0$



Bifurcation from simple eigenvalue $(\lambda_1, 0)$

Proposition 1(Crandall and Rabinowitz (1971), Rabinowitz (1971))

Assume $\int_\Omega m dx < 0$. Then, there exists a subcontinuum $\mathcal{C}$ (maximal, closed, and connected subset of $\mathbb{R} \times C(\overline{\Omega})$) of positive solutions bifurcating at $(\lambda_1, 0)$ and it is

$$\textbf{subcritical} \quad \textbf{if} \int_{\partial\Omega} \phi_1^{p+1} ds \geq \int_\Omega \phi_1^{p+1} dx,$$
$$\textbf{supercritical if} \int_{\partial\Omega} \phi_1^{p+1} ds < \int_\Omega \phi_1^{p+1} dx.$$

Here $\phi_1 := \varphi_1(\lambda_1)$ is a positive principal eigenfunction to $\mu_1(\lambda_1) = 0$.

主結果4 Characterization (subcritical case)

Theorem 4

Assume $\int_\Omega m dx < 0$ **and the bifurcation** $\mathcal{C}$ **at** $(\lambda_1, 0)$ **is subcritical. Then,** $\mathcal{C}$ **is unbounded in** $\mathbb{R} \times C(\overline{\Omega})$ **and the following assertions hold true:**

(a) If we set $J := \{\lambda > 0 : (\lambda, u) \in \mathcal{C}\}$**, then we have** $J = (0, \lambda_1)$**. Bifurcation from infinity is possible only at** $\lambda = 0$ **if it occurs.**

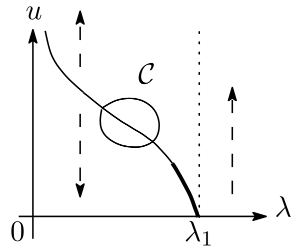
(b) Positive solutions for $0 < \lambda < \lambda_1$ **are all unstable.**

(c) There is no positive solution for $\lambda = \lambda_1$ **and**

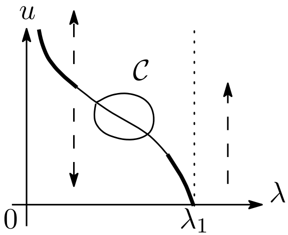
$$\Lambda = \sup\{\lambda > 0 : \exists \text{ positive solution for } \lambda\} < \infty.$$

We have $\Lambda = \lambda_1$ **provided that** $m(x) \leq 0$ **on** $\partial\Omega$**.**

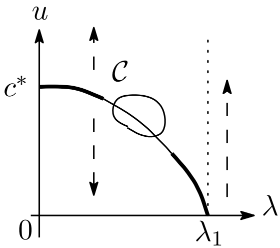
Remark. **see Pflüger (1999).**



$|\Omega| > |\partial\Omega|$



$|\Omega| = |\partial\Omega|$,
$N = 2, 3$, **and** $p = 2$



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主結果5 Characterization (supercritical case)

Theorem 5

Assume $\int_\Omega m dx < 0$ **and the bifurcation** $\mathcal{C}$ **at** $(\lambda_1, 0)$ **is supercritical. Additionally if** $m(x) \le 0$ **on** $\partial\Omega$**, then** $\mathcal{C}$ **is extended to the whole** $\lambda > \lambda_1$ **and parametrized by** $\lambda > \lambda_1$**, and thus there exists at least one positive solution for** $\lambda > \lambda_1$**.**

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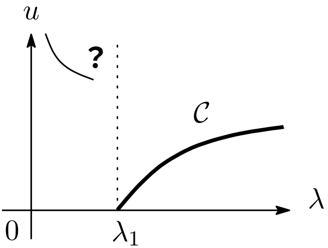

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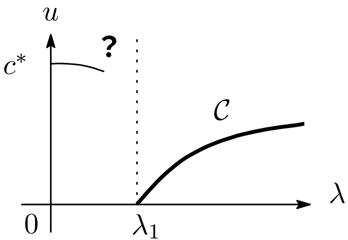
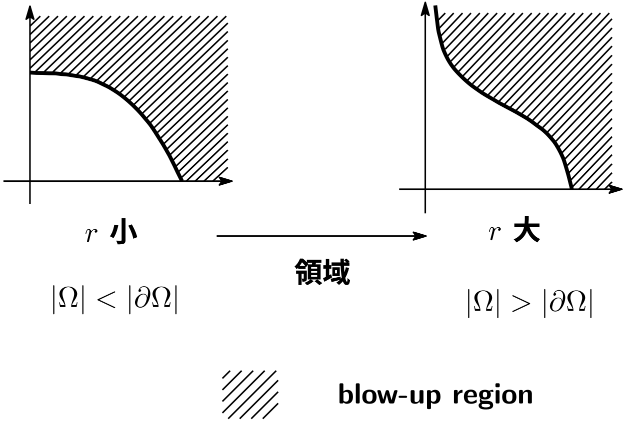


$|\Omega| < |\partial\Omega|$

Blow-up region の考察 ($\int_\Omega m dx < 0$)

$\Omega \subset \mathbb{R}^2$, **a disk with radius** $r > 0$



An example m in the subcritical case

Consider the existence of ϕ_1 of the problem

$$\begin{cases} -\Delta\phi_1 = \lambda_1 m \phi_1 & \textbf{in } \Omega, \\ \frac{\partial \phi_1}{\partial \mathbf{n}} = 0 & \textbf{on } \partial\Omega \end{cases}$$

with the following condition.

$$\int_\Omega \phi_1^{p+1} dx < \int_{\partial\Omega} \phi_1^{p+1} ds \textbf{ and } |\Omega| \geq |\partial\Omega|, \textbf{ or}$$

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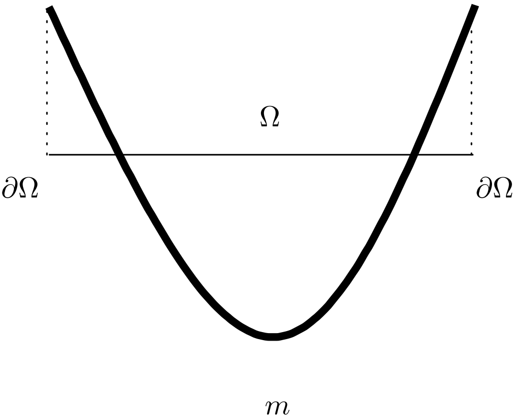
Example (Kurata)

For this it is sufficient to construct m such that
$\dfrac{\int_\Omega \phi_1^{p+1} dx}{\int_{\partial\Omega} \phi_1^{p+1} ds} < \dfrac{|\Omega|}{|\partial\Omega|}$. **We can do it.**

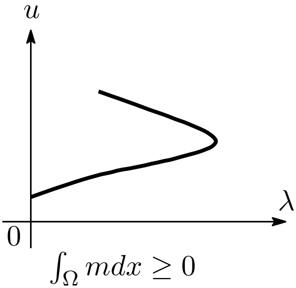
Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$, and let m radially symmetric such that $\int_\Omega m dx < 0$ and

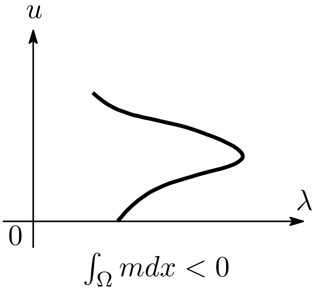
$$m \begin{cases} > 0, & \frac{1}{2} < |x|, \\ < 0, & |x| < \frac{1}{2}. \end{cases}$$



Future problems

sign-indefinite な m に対して，supercritical な bifurcation component の turning point の存在





- **境界値問題**

$$\begin{cases} -\Delta v = \lambda m(x) v - v^p & \textbf{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = v^p & \textbf{on } \partial\Omega. \end{cases}$$

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$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v - v^p & \text{in } (0, \infty) \times \Omega, \\ v(0, x) = \psi_0(x) & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = v^p & \text{on } (0, \infty) \times \partial\Omega. \end{cases}$$

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- blow-up argument of the initial-boundary value problem for large ψ_0

PROOF

A priori upper bounds

Consider

$$\begin{cases} -\Delta u = f(x,u) & \textbf{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = g(x,u) & \textbf{on } \partial\Omega, \end{cases}$$

where $f \in C(\overline{\Omega} \times [0,\infty))$, $g \in C^{1+\theta}(\partial\Omega \times [0,\infty))$, **and**

$$\lim_{t\to\infty} \frac{f(x,t)}{t^p} = h(x) \textbf{ uniformly in } \overline{\Omega},$$

$$\lim_{t\to\infty} \frac{g(x,t)}{t^r} = i(x) \textbf{ uniformly on } \partial\Omega \textbf{ with } i > 0$$

Here $1 < p < \frac{N+2}{N-2}$, $1 < r < \frac{N}{N-2}$, **and** $p < 2r - 1$.

Theorem(Morales-Rodrigo and Suárez(2005))

Let $u \in C^2(\overline{\Omega})$ **be a nonnegative solution of the problem that attains the maximum on** $\partial\Omega$**. Then, there exists a constant** $C = C(p, r, \|h\|_\infty, \|i\|_{\infty,\partial\Omega}) > 0$ **such that** $u(x) \leq C$ **in** $\overline{\Omega}$**.**

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$I = [a, b]$, $-\infty < a < b < \infty$,

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$$\Downarrow \ u = \lambda^{-1/(p-1)} v \ (\lambda > 0)$$

$$\begin{cases} -\Delta u = \lambda(mu - u^p) & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u^p & \text{on } \partial\Omega. \end{cases} \Longrightarrow \text{If } \lambda \in I \subset (0, \infty), \text{ then } u(x) \le C(I) \text{ in } \overline{\Omega}.$$

Remark (Case $p \geq 2r-1$). **Under some hypotheses, for $1 < r < r_0(N)(< \frac{3}{2})$ it can be proved that for any $b > 0$ there exists a constant $C_b > 0$ such that any positive solution (λ, u), $\lambda \in (0, b]$, of the problem**

$$\begin{cases} -\Delta u = \lambda(mu - u^2) & \textbf{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u^r & \textbf{on } \partial\Omega, \end{cases}$$

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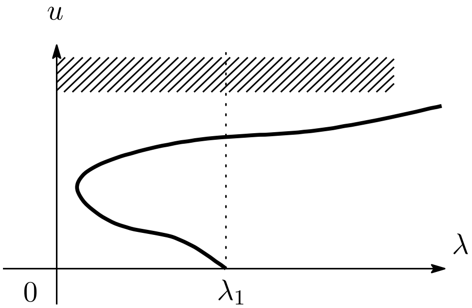

Critical case $p = 2r - 1$

$p < 2r - 1 \implies$ **Blow-up argument**

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$p < 2r - 1 \implies$ **Blow-up argument**
Consider

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u) & \textbf{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) \geq 0 & \textbf{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = g(u) & \textbf{on } (0,\infty) \times \partial\Omega. \end{cases}$$

Here $f(u) \sim -u^p$**,** $g(u) \sim u^r$**,** $u \to \infty$**.**

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u) - \int_{\partial\Omega} G(u) \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u) \\ &\quad + \frac{|\partial\Omega|}{|\Omega|}\int_\Omega \left\{G(u) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} G(u)\right\} - \frac{|\partial\Omega|}{|\Omega|}\int_\Omega G(u). \end{aligned}$$

$$\int_\Omega |\nabla G(u)| = \int_\Omega |g(u)||\nabla u| \le \varepsilon \int_\Omega |\nabla u|^2 + \frac{C}{\varepsilon}\int_\Omega |g(u)|^2$$

$$\therefore \quad E(u) \ge \left(\frac{1}{2} - \varepsilon\right)\int_\Omega |\nabla u|^2 - \int_\Omega F(u) - \frac{C}{\varepsilon}\int_\Omega |g(u)|^2 - \frac{|\partial\Omega|}{|\Omega|}\int_\Omega G(u)$$

$-F(u) \sim u^{p+1}$, $|g(u)|^2 \sim u^{2r}$

$$\Longrightarrow \begin{cases} p+1 > 2r & \Longrightarrow \textbf{Global existence} \\ p+1 < 2r & \Longrightarrow \textbf{Blow-up} \end{cases}$$

(see Rodríguez-Bernal and Tajdine (2001))

Stability and instability arguments

positive solution (λ, u) **における線形化固有値問題：** $\mu = \mu(\lambda, u)$,

$$\begin{cases} -\Delta\varphi = \lambda(m\varphi - pu^{p-1}\varphi) + \mu\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\mathbf{n}} = \lambda p u^{p-1}\varphi & \text{on } \partial\Omega. \end{cases}$$

最小固有値 $\mu_1(\lambda, u)$ **について**

Proposition 2 (Nondegeneracy)

(a) If $0 < \lambda \leq \lambda_1$**, then** $\mu_1(\lambda, u) \neq 0$ **for all positive solution** (λ, u)**.**
(b) For each $\lambda > \lambda_1$ **we have** $\mu_1(\lambda, u) \neq 0$ **for all positive solution** (λ, u) **provided that** $m(x) \leq 0$ **on** $\partial\Omega$**.**

Here it is understood that $\lambda_1 = 0$ **when** $\int_\Omega m\,dx \geq 0$**.**

Proof of (a). **positive solution** (λ, u) **に対して**

$$\int_\Omega (|\nabla u|^2 - \lambda m u^2) + \lambda \left(\int_\Omega u^{p+1} - \int_{\partial\Omega} u^{p+1} \right) = 0.$$

$$0 \le \mu_1(\lambda, 0) = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^2 - \lambda m u^2}{\int_\Omega u^2}.$$

$$\therefore \int_{\partial\Omega} u^{p+1} ds \ge \int_\Omega u^{p+1} dx.$$

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$$\therefore \int_{\partial\Omega} u^{p+1} ds \ge \int_\Omega u^{p+1} dx.$$

さて，$\exists(\lambda, u)$ positive solution s.t. $\mu_1(\lambda, u) = 0$ ならば

$$\begin{cases} -\Delta\varphi_1 = \lambda(m\varphi_1 - p u^{p-1}\varphi_1) & \text{in } \Omega, \\ \frac{\partial\varphi_1}{\partial\mathbf{n}} = \lambda p u^{p-1}\varphi_1 & \text{on } \partial\Omega. \end{cases}$$

Picone の恒等式による考えにより

$$\left(\frac{u}{\varphi_1}\right)(\Delta u \varphi_1 - u \Delta\varphi_1) = -\lambda u^{p+1}.$$

ここで

$$\Delta u\varphi_1 - u\Delta\varphi_1 = \sum_j \frac{\partial}{\partial x_j}\left(\varphi_1^2\left(\frac{\partial}{\partial x_j}\left(\frac{u}{\varphi_1}\right)\right)\right)$$

に注意して，Green の公式を用いて

$$\int_\Omega \left(\frac{u}{\varphi_1}\right)\sum_j \frac{\partial}{\partial x_j}\left(\varphi_1^2\left(\frac{\partial}{\partial x_j}\left(\frac{u}{\varphi_1}\right)\right)\right) = -\lambda\int_\Omega u^{p+1}.$$

$$\therefore\ -\int_\Omega \varphi_1^2\left|\nabla\left(\frac{u}{\varphi_1}\right)\right|^2 - \lambda\int_{\partial\Omega} u^{p+1} = -\lambda\int_\Omega u^{p+1}.$$

$$\therefore\ \int_{\partial\Omega} u^{p+1} < \int_\Omega u^{p+1}.$$ **矛盾**

Proof of (b). $\exists$ positive solution s.t. $\mu_1(\lambda, u) = 0$ ならば
$f(u) = mu - u^p$, $g(u) = u^p$ に対して

$$\begin{cases} -\Delta\varphi_1 = \lambda f'(u)\varphi_1 & \text{in } \Omega, \\ \frac{\partial\varphi_1}{\partial\mathbf{n}} = \lambda g'(u)\varphi_1 & \text{on } \partial\Omega. \end{cases}$$

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial\mathbf{n}} = \lambda g(u) & \text{on } \partial\Omega. \end{cases}$$

$$\therefore \int_\Omega -\Delta u f'(u)\varphi_1 + f(u)\Delta\varphi_1 = 0.$$

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Green の公式より

$$\int_\Omega -\Delta u f'(u)\phi_1 + f(u)\Delta\phi_1$$

$$= \int_\Omega f''(u)|\nabla u|^2\varphi_1 - \lambda\int_{\partial\Omega}(g(u)f'(u) - g'(u)f(u))\varphi_1.$$

$$f''(u) = -p(p-1)u^{p-2},$$
$$g(u)f'(u) - g'(u)f(u) = -(p-1)m(x)u^p.$$

$$\therefore\ \ 0 = \int_\Omega f''(u)|\nabla u|^2\varphi_1 + \lambda(p-1)\int_{\partial\Omega} m(x)u^p\varphi_1.$$

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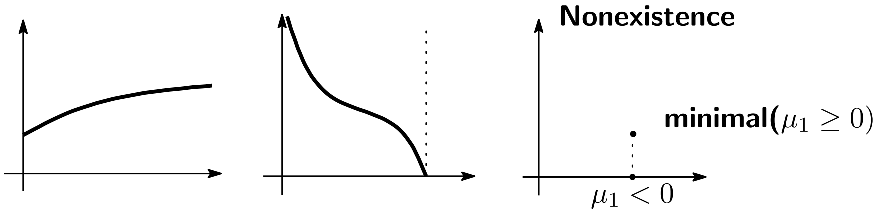

No positive solutions for any $\lambda \gg 1$

Consider the reduced problem

$$\begin{cases} -\Delta v = \lambda m(x) v - v^p & \textbf{in } \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = v^p & \textbf{on } \partial\Omega. \end{cases}$$

Theorem 6

Assume $m \geq 0$ in $\overline{\Omega}$. Then the problem has no positive solution for any $\lambda \gg 1$ provided that

$$m(x) = m_0 > 0 \textbf{ in } \Omega_\delta,$$
$$\max_{x \in \overline{\Omega}} m(x) = m_0.$$

Outline of the proof. '$\exists\lambda_n \nearrow \infty$ **with positive solution** (λ_n, v_n)' **と仮定する.**

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Fix $0 < \delta' < \delta$. **Then,** $\min_{x \in \overline{\Omega_{\delta'}}} v_n(x) \to \infty, \ n \to \infty$.

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Since $m \geq 0$, we note

$$\begin{cases} \frac{\partial v_n}{\partial t} \geq \Delta v_n - v_n^p & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial v_n}{\partial \mathbf{n}} = v_n^p & \text{on } (0,\infty) \times \partial\Omega. \end{cases}$$

Theorem(Arrieta and Rodríguez-Bernal(2004))

Let $x_0 \in \partial\Omega$ **and** $\rho_0 > 0$**. Then, there exist** $0 < \rho < \rho_0$**,** $T > 0$**, a positive smooth function** $v(t,x)$ **defined for** $(t,x) \in [0,T) \times (\Omega \cap B(x_0;\rho))$**, and an initial data** $\psi_0 \in C_0^\infty(B(x_0;\rho_0))$ **such that as** $t \nearrow T$**,**

$$v(t,x) \nearrow v(T,x) = \frac{C}{\mathrm{dist}(x,\partial\Omega)^{2/(p-1)}}, \quad x \in \Omega \cap B(x_0;\rho),$$

and that if $w(t,x;w_0)$ **is a unique solution of the initial-boundary value problem with initial data** $w_0 = \psi_0|_\Omega$**, then**

$$v(t,x) \le w(t,x;w_0), \quad (t,x) \in [0,T) \times (\Omega \cap B(x_0;\rho)).$$

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$$v(t,x) \le w(t,x;w_0), \quad (t,x) \in [0,T) \times (\Omega \cap B(x_0;\rho)).$$

We notice from Proposition 3 that $v_n \ge \psi_0|_\Omega$ **if** $n \gg 1$**.**

Uniformly blow-up near the boundary

Proposition 3 を示すにはノイマン問題を考えれば十分．

$$\begin{cases} -\Delta w = \lambda m(x) w - w^p & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

の unique positive solution (λ_n, w_n) について

$$\min_{\overline{\Omega_{\delta'}}} w_n \to \infty, \ \ n \to \infty$$

を示す．

Remark. $m > 0$ **in** $\overline{\Omega}$ **ならば容易．** $\underline{w} = \lambda^{1/(p-1)} \underline{m}^{1/(p-1)}$ **は subsolution であるから．ただし，** $\underline{m} = \min_{\overline{\Omega}} m$**．**

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Lemma 1

For a compact set $K \subset \{x \in \Omega : m(x) > 0\}$**, we have** $\min_K w_n \to \infty$ **as** $n \to \infty$**.**

Outline of Proof of Proposition 3. contradiction argument による. $n \geq 1$,

$$x_n \in \overline{\Omega_{\delta'}}, \quad w_n(x_n) = \min_{\overline{\Omega_{\delta'}}} w_n \leq C_0$$

とせよ．このとき， $x_n \in \partial\Omega$ **を導く．**

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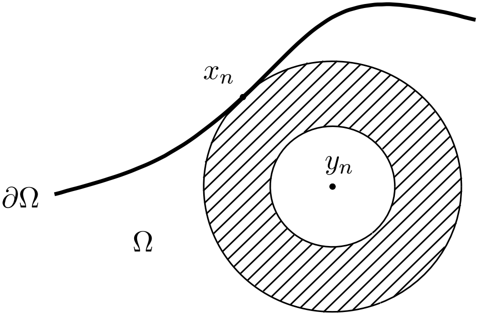
これより，

$$-\Delta w_n = \lambda_n m(x) w_n - w_n^p = \lambda_n m_0 w_n - w_n^p \geq 0 \quad \text{in } \Omega_\delta.$$

強最大値の原理から w_n は Ω_δ の境界で最小値を取る．ところが，Lemma 1 によって $x_n \in \partial\Omega$ が従う．

最後に Du (2005) の idea に従って証明を終える．



$$w_n(x) \geq \eta_n(x) := w_n(x_n) + \psi(|x - y_n|),$$

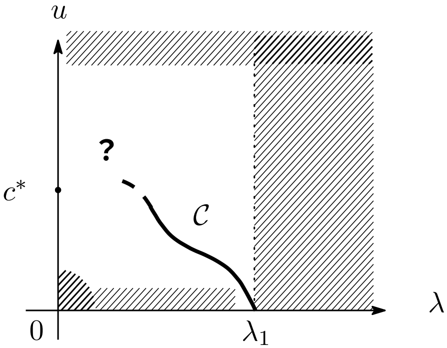
$\psi(t) = e^{-t^2} - e^{-R^2}$ **(**$\frac{R}{2} \leq t \leq R$**).**

$\nu_n = \frac{y_n - x_n}{|y_n - x_n|}$,

$$\frac{\partial \eta_n}{\partial \nu_n}(x_n) = \psi'(R)(-1) = 2Re^{-R^2} > 0$$ **矛盾**

Bifurcation in the subcritical case

Consider the case $\int_\Omega m dx < 0$**.**



subcritical case

Determine the behavior of the biufurcation component $\mathcal{C}$ **in the subcritical case.**

$$\begin{cases} -\Delta u = \lambda(mu - u^p) & \textbf{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u^p & \textbf{on } \partial\Omega \end{cases} \overset{v=\lambda^{1/(p-1)}u}{\Longrightarrow} \begin{cases} -\Delta v = \lambda m v - v^p & \textbf{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = v^p & \textbf{on } \partial\Omega. \end{cases}$$

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$\lambda \simeq 0$,

$$v = \lambda^{1/(p-1)} \left(\frac{\int_\Omega m dx}{|\Omega| - |\partial\Omega|} \right)^{1/(p-1)} + \textbf{ higher order terms w.r.t. } \lambda$$

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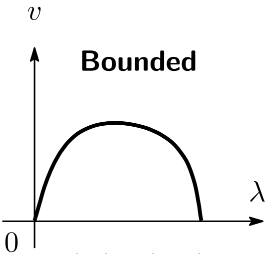
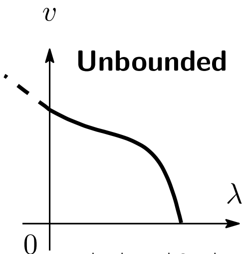


$|\Omega| < |\partial\Omega|$



$|\Omega| > |\partial\Omega|$

Operator equations

$$v = K_\Omega[f] \qquad \begin{cases} (-\Delta + 1)v = f(x) & \textbf{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \textbf{on } \partial\Omega, \end{cases}$$

$$v = K_{\partial\Omega}[g] \qquad \begin{cases} (-\Delta + 1)v = 0 & \textbf{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = g(x) & \textbf{on } \partial\Omega. \end{cases}$$

Operator equations

$$\begin{cases} -\Delta v = \lambda m v - |v|^{p-1}v & \textbf{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = |v|^{p-1}v & \textbf{on } \partial\Omega, \end{cases}$$

$$\iff \quad 0 = v - K_\Omega[(\lambda m + 1)v - |v|^{p-1}v] - K_{\partial\Omega}[|v|^{p-1}v]$$
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$$F : \mathbb{R} \times C(\overline{\Omega}) \longrightarrow C(\overline{\Omega}),$$

$F_\lambda, F_v,$ **and** $F_{\lambda v}$ **are continuous at** $(0,0)$,

$$F_v(0,0)\phi = \phi - K_\Omega[(\lambda m + 1)\phi,$$

$$\mathrm{Ker} F_v(0,0) = \langle 1 \rangle,$$

$$F_{\lambda v}(0,0)[1] = -K_\Omega[m] \notin \mathrm{Im} F_v(0,0)$$

Crandall and Rabinowitz

$$(\lambda, v) = (\lambda(s),\ s(1 + z(s))), \quad |s| < \varepsilon$$

with $\lambda(0) = 0$ **and** $z(0) = 0$**.**

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$$\therefore \lim_{s\to+0} \frac{\lambda}{s^{p-1}} = \frac{|\Omega| - |\partial\Omega|}{\int_\Omega m dx}.$$

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$$\therefore \lim_{s \to +0} \frac{\lambda}{s^{p-1}} = \frac{|\Omega| - |\partial\Omega|}{\int_\Omega m dx}.$$

$$s \sim \lambda^{1/(p-1)} \left(\frac{\int_\Omega m dx}{|\Omega| - |\partial\Omega|} \right)^{1/(p-1)}, \quad 0 < s \ll 1$$

ご静聴ありがとうございました.