Positivity of bifurcating nontrivial nonnegative solutions of indefinite concave problems

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Our purpose

In this talk, we discuss positivity of nontrivial nonnegative solutions of the concave problem

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here, $a \in C(\overline{\Omega})$ changes sign and $q \in (0,1)$.

$\surd$ The strong maximum principle or Hopf's lemma *does not work* for obtaining that each nontrivial nonnegative solution of $(\mathcal{P})$ implies an interior point of the positive cone $\{u \in C^1(\overline{\Omega}) : u \geq 0\}$.

$\surd$ We investigate such q's set of $(0,1)$ by a bifurcation approach, the sub and supersolution method and a variational technique.

Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider nontrivial nonnegative solutions, say *positive solutions* $u > 0$, of the problem

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $a \in C(\overline{\Omega})$ changes sign, $q \in (0,1)$ and ν is the unit outer normal to $\partial\Omega$.

We say a *strongly positive solution* $u \gg 0$ if $u > 0$ in $\overline{\Omega}$. Then,

Q: For which $q \in (0,1)$, can we deduce $u \gg 0$ from solution $u > 0$?

We can do it for all $q \geq 1$. We shall see that this is also possible for $q \in (0,1)$ close to 1.

A counterexample ($\exists$ sol. $u > 0$ but *not* $\gg 0$)

Let $q \in (0,1)$. Define $\Omega := (0,\pi)$,

$$r := \frac{2}{1-q} \in (2,\infty), \quad \text{and} \quad a(x) := r^{1-\frac{2}{r}} \left(1 - r\cos^2 x\right).$$

We set

$$u(x) := \frac{\sin^r x}{r} > 0, \quad \text{and it does } \textit{not} \text{ satisfy } u \gg 0.$$

Moreover, u satisfies

$$\begin{cases} -u''(x) = a(x)u(x)^q & \text{in} \quad x \in (0,\pi), \\ u(x) = u'(x) = 0 & \text{at} \quad x = 0, \pi \end{cases}$$

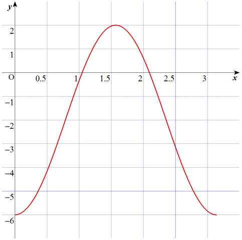


Figure: The weight a with $q = \frac{1}{2}$.

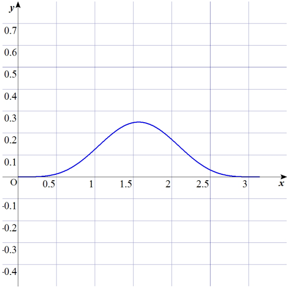


Figure: The solution $u > 0$ in the case $q = \frac{1}{2}$.

Pioneer work

Bandle, Pozio and Tesei (Math. Z., 1988) proved:

- If $\int_\Omega a < 0$, then $(\mathcal{P})$ has *at least* one solution $u > 0$.
- $(\mathcal{P})$ has a solution $u > 0$ which is positive in Ω_+ iff $\int_\Omega a < 0$.
- $(\mathcal{P})$ has *at most* one solution $u \gg 0$,

where

$$\Omega_+ := \{x \in \Omega : a(x) > 0\}.$$

We first prove the existence of a solution $u \gg 0$ of $(\mathcal{P})$.

Trivial line

Consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \mu a(x)\phi & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\int_\Omega a < 0$, there exists a unique positive principal eigenvalue $\mu_1(a)$, which is simple and possesses the eigenfunction $\phi_1 = \phi_1(a) \gg 0$ such that $\int_\Omega \phi_1^2 = 1$.

So, when $\mu_1(a) = 1$, $(\mathcal{P})$ has the trivial line $(q, u) = (1, t\phi_1)$:

$$\begin{cases} -\Delta\phi_1 = a(x)\phi_1 & \text{in } \Omega, \\ \frac{\partial \phi_1}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Main result 1

Put

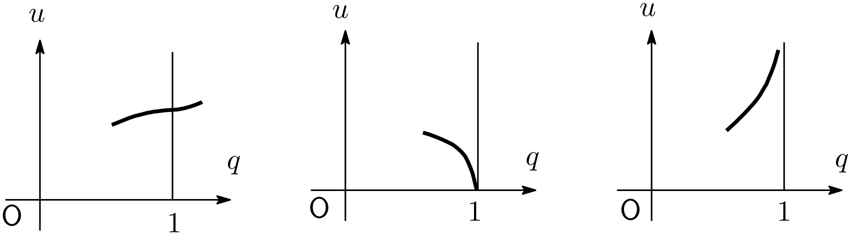
$$t_* := \exp\left[-\frac{\int_\Omega a(x)\phi_1^2 \log\phi_1}{\int_\Omega a(x)\phi_1^2}\right].$$

Theorem 1. *Assume* $\int_\Omega a < 0$. *Then,* $(\mathcal{P})$ *has a solution* $u_q \gg 0$ *for* $q \in (q_0, 1)$ *for some* $q_0 \in [0, 1)$. *Moreover, as* $q \to 1^-$,

$$\begin{cases} u_q \longrightarrow t_*\phi_1 \quad \text{in } C^1(\overline{\Omega}) & \text{if } \mu_1(a) = 1, \\ \|u_q\|_{C^1(\overline{\Omega})} \longrightarrow 0 & \text{if } \mu_1(a) > 1, \\ \min_{\overline{\Omega}} u_q \longrightarrow \infty & \text{if } \mu_1(a) < 1. \end{cases}$$

Corollary 2. *Let* q_0 *as in Theorem 1, and let* $q \in (q_0, 1)$. *Then,* $(\mathcal{P})$ *has a solution* $u \gg 0$ *iff* $\int_\Omega a < 0$.

Local bifurcation diagrams



$\mu_1(a) = 1$ $\mu_1(a) > 1$ $\mu_1(a) < 1$

We next investigate the global structure of the positive solutions set $\{(q,u) : u > 0\}$.

q's sets $\mathcal{A}, \mathcal{I}$

Under $\int_\Omega a < 0$, introduce

$$\mathcal{A} := \{q \in (0,1) : u \gg 0 \text{ for any solution } u > 0 \text{ of } (\mathcal{P})\},$$

$$\mathcal{I} := \{q \in (0,1) : (\mathcal{P}) \text{ has a solution } u \gg 0\} \quad (\implies (q_0, 1) \subseteq \mathcal{I}\ )$$

Then, we see $\mathcal{A} \subseteq \mathcal{I}$, since $(\mathcal{P})$ has a solution $u > 0$ for every $q \in (0,1)$ (BPT), and $q \in \mathcal{A}$ implies $u \gg 0$. In particular, $q \in \mathcal{A}$ implies that a solution $u > 0$ is unique.

Q: Can we characterize the sets $\mathcal{A}, \mathcal{I}$?

Let us introduce the condition

$$(A.1) \qquad \Omega_+ = \bigcup_{k=1}^m \Omega_{+,k}, \text{ where } \Omega_{+,k} \text{ is connected.}$$

Main result 2

Theorem 3. *Assume* $\int_\Omega a < 0$. *Then, the following three assertions hold:*

(i) $\mathcal{I} = (q_1, 1)$ *for some* $q_1 \in [0, q_0]$.

(ii) *Assume additionally* $(A.1)$. *Then,*

$$\mathcal{A} = (q_2, 1) \quad \textit{for some} \quad q_2 \in [q_1, 1).$$

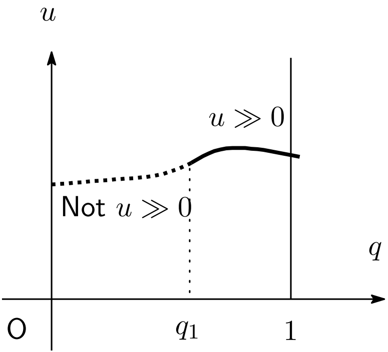
(iii) *Assume additionally that* Ω_+ *is connected and* $\partial\Omega_+$ *is smooth. Then*

$$\mathcal{A} = \mathcal{I}, \quad \textit{i.e.}, \quad q_1 = q_2.$$

The next figure may illustrate the situation (iii):

Global bifurcation diagrams 1

Case Ω_+ is connected and $\partial\Omega_+$ is smooth (single component case):



$\mu_1(a) = 1$

$\mathcal{A} = \mathcal{I}$

Indeed, since a solution $u > 0$ satisfies $u > 0$ in Ω_+, it is unique for *every* $q \in (0, 1)$, which follows from the uniqueness result by BPT:

Positivity and uniqueness results from BPT (Math. Z., 1988)

- For each k, any solution $u > 0$ of $(\mathcal{P})$ satisfies

$$\text{either} \quad u \equiv 0 \quad \text{or} \quad u > 0 \quad \text{in} \quad \Omega_{+,k},$$

(Lemma 2.2).

- Assume $(A.1)$ holds and $\partial\Omega_+$ is smooth. Then, for any $J \subset \{1, 2, 3, \dots, m\}$, $(\mathcal{P})$ has *at most* one solution $u > 0$ which satisfies

$$u > 0 \quad \text{in} \quad \Omega_{+,J} := \bigcup_{k \in J} \Omega_{+,k},$$
$$u \equiv 0 \quad \text{in} \quad \Omega_+ \setminus \Omega_{+,J},$$

(Theorem 3.1).

Asymptotic behavior as $q \to 0^-$

Lemma 4. *Assume* $\int_\Omega a < 0$ *and* $(A.1)$*. Then we have* $C, \overline{q} \in (0,1)$ *such that*

$$C \leq \|u\|_\infty \leq C^{-1}$$

for all solution $u > 0$ *of* $(\mathcal{P})$ *for* $q \in (0, \overline{q})$*.*

Under $\int_\Omega a < 0$ and $(A.1)$, we take a solution $u_n > 0$ of $(\mathcal{P})$ for $q = q_n \to 0^+$. By elliptic regularity, we have, up to a subsequence, $u_n \to u_0 \geq 0$ in $C^1(\overline{\Omega})$, and Lemma 4 shows $u_0 > 0$. Then, we obtain

- u_0 is *not* a solution of the limiting problem $(\mathcal{P})$ with $q = 0$.
- u_0 possesses a dead core D_0 with positive measure in Ω.
- If we additionally assume Ω_+ is connected, then $u_0 > 0$ in Ω_+ and so, $D_0 \subset \Omega \setminus \Omega_+$.

Sketch of proof of Theorem 3 (ii)

Under $(A.1)$ we prove $\mathcal{A} = (q_2, 1)$. Let Ω' be a subdomain of Ω_+. Consider the smallest eigenvalue $\lambda_1(a, \Omega')$ of

$$-\Delta\varphi = \lambda a^+ \varphi \ \text{ in } \Omega', \quad \varphi|_{\partial\Omega'} = 0,$$

and let φ_1 be the corresponding positive eigenfunction such that $\|\varphi_1\|_\infty = 1$.

Lemma 5. *Assume a domain $D \Subset \Omega'$ and $\lambda_1(a, D) < 1$. Then, $u \geq \varphi_1$ in D for all nontrivial functions $u \geq 0$ such that $-\Delta u = au^q$ in Ω' with $q \in (0, 1)$.*

Lemma 6. *Assume $(A.1)$, and $\lambda_1(a, \Omega_{+,k}) < 1$ for any $k = 1, 2, \ldots, m$. Then we have $\|u\|_{H^1(\Omega)} \geq C$ for all solutions $u > 0$ of $(\mathcal{P})$ with $q \in (0, 1)$.*

Remark 7. By replacing a by ca with $c > 0$ large, the condition $\lambda_1(a, \Omega_{+,k}) < 1$ is achieved.

(continued)

Assume $q_n \to 1^-$ and a solution $u_n > 0$ of $(\mathcal{P})$ for $q = q_n$ such that it does *not* satisfy $u_n \gg 0$. If $\{u_n\}$ is bounded in $H^1(\Omega)$, then up to a subsequence, $u_n \rightharpoonup u_0$ in $H^1(\Omega)$, $u_n \to u_0$ in $L^p(\Omega)$ with $p < 2^*$, and $u_n \to u_0$ a.e. From

$$\int_\Omega \nabla u_n \nabla (u_n - u_0) = \int_\Omega a(x)\, u_n^{q_n} (u_n - u_0) \longrightarrow 0,$$

we infer that $u_n \to u_0$ in $H^1(\Omega)$, and thus, $u_n \to u_0$ in $C^1(\overline{\Omega})$. By Lemma 6, we have $u_0 > 0$, which is a positive solution of

$$-\Delta u_0 = a u_0 \ \text{ in } \Omega, \quad \frac{\partial u_0}{\partial \nu} = 0 \ \text{ on } \partial\Omega.$$

We now get $u_0 \gg 0$, and then, $u_n \gg 0$ for n large, a contradiction.

If $\{u_n\}$ is unbounded, then we put $v_n := u_n / \|u_n\|_{H^1(\Omega)}$, and then, a similar argument is carried out to deduce a contradiction.

(continued)

We have obtained $(q,1) \subset \mathcal{A}$ and can prove $\mathcal{A}$ is open in a similar argument.

Finally, we prove $\mathcal{A}$ is connected.

Lemma 8. *If $q_0 \in \mathcal{A}$ then $\left(q_0,\ \frac{1}{2-q_0}\right) \subseteq \mathcal{A}$.*

Based on Lemma 8, the following iteration scheme is employed: Set q_n such that

$$(q_{n-1},\ q_n] \subseteq \left(q_{n-1}, \frac{1}{2-q_{n-1}}\right) \subseteq \mathcal{A},$$

$$q_n := \frac{1}{2-q_{n-1}} - \frac{1}{10}\left(\frac{1}{2-q_{n-1}} - q_{n-1}\right), \quad n \geq 1.$$

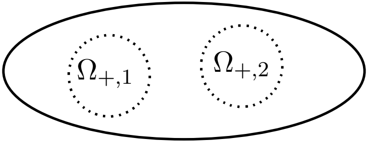
Since $q_n \nearrow q \leq 1$, passing to the limit we deduce

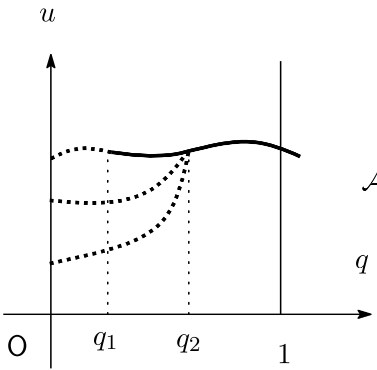
$$q = \frac{1}{2-q} - \frac{1}{10}\left(\frac{1}{2-q} - q\right),$$

$$\therefore \quad q = 1, \quad \text{as desired.}$$

Global bifurcation diagrams 2 (expectation)

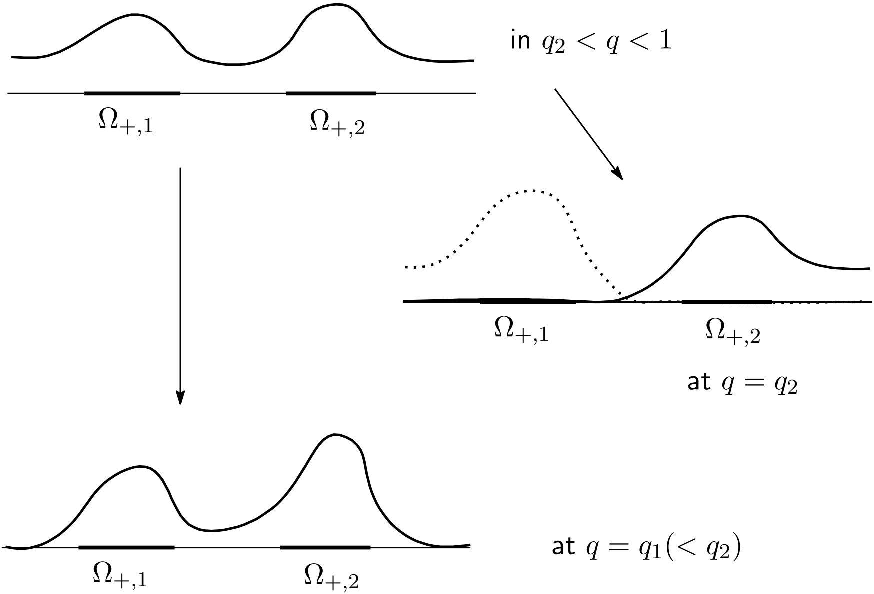
Case $\Omega_+ = \Omega_{+,1} \cup \Omega_{+,2}$ (two component case):





$\mu_1(a) = 1$

$\mathcal{A} \subsetneq \mathcal{I} \subsetneq (0,1)$


$\Omega_{+,1}$
$\Omega_{+,2}$
in $q_2 < q < 1$
$\Omega_{+,1}$
$\Omega_{+,2}$
at $q = q_2$
$\Omega_{+,1}$
$\Omega_{+,2}$
at $q = q_1 (< q_2)$

Remarks on $\mathcal{A}, \mathcal{I}$

Remark 9.

(i) There exists $a \in C(\overline{\Omega})$ such that $\mathcal{A} \subsetneq \mathcal{I}$. More precisely, let $\Omega := (x_0, x_1) \subset \mathbb{R}$. Then, for any $q \in (0,1)$ there exists $a \in C(\overline{\Omega})$ such that $q \in \mathcal{I} \setminus \mathcal{A}$.

(ii) Consider

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} -\Delta u = (a(x) - \varepsilon)u^q & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\int_\Omega a = 0$ (consequently, $a - \varepsilon$ changes sign and $\int_\Omega (a - \varepsilon) < 0$). Then, for $\varepsilon \to 0^+$ we can choose $q_\varepsilon \to 0^+$ such that $\mathcal{I}_\varepsilon = (q_\varepsilon, 1)$.

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Thank you for your kind attention.