An exact multiplicity result for some sublinear Robin problem with an indefinite weight

Kenichiro Umezu (Ibaraki Univ.)

joint work with
Uriel Kaufmann (Univ. Nacional de Córdoba)
Humberto Ramos Quoirin (Univ. Nacional de Córdoba)

SS3, AIMS2023Wilmington

June 1, 2023

Problem

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with a smooth boundary $\partial\Omega$. Consider the **sublinear Robin** problem

$$(P_\alpha) \qquad \begin{cases} -\Delta u = a(x)\, u^q & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ \partial_\nu u = \alpha u & \text{on} \quad \partial\Omega. \end{cases}$$

Here:

- Δ is the usual Laplacian.
- $a \in C^\theta(\overline{\Omega})$ with $0 < \theta < 1$ **changes sign**.
- ν is the unit outer normal to $\partial\Omega$, and $\partial_\nu u = \frac{\partial u}{\partial \nu}$.

Our purpose is, given $\mathbf{0 < q < 1}$, to understand the **positive solution set** $\{(\alpha, u)\}$ of (P_α) for $\boldsymbol{\alpha} \geq \mathbf{0}$.

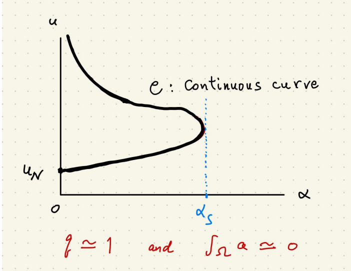
- u is a **nontrivial solution** of $(P_\alpha) \overset{\text{def}}{\Longleftrightarrow} 0 \not\equiv u \in C^{2+\tau}(\overline{\Omega})$ admits (P_α).
- A nontrivial solution $u \in P^\circ := \{u \in C(\overline{\Omega}) : u > 0 \text{ in } \overline{\Omega}\}$ is said to be a **positive solution**.

Expected positive solution set

- Contrast:
  - A nontrivial solution is positive when $q \geq 1$;
  - when $0 < q < 1$, we can construct some nontrivial solution **that vanishes in a subdomain of Ω** ([KRQU, *NoDEA*, 2018] for $\alpha = 0$); [Friedman, Phillips, *Trans. AMS*, 1984], [García-Melián, Rossi, Sabina de Lis, *Proc. LMS*, 2007].

• Condition $\int_\Omega a(x) < 0$, say **(*A*.0)**, is **necessary** for the existence of a positive solution for $\alpha \geq 0$.

• Under some additional conditions to $(A.0)$, we will deduce the **global exact low multiplicity** of positive solutions:



start with Neumann case (P_0) →

Neumann case (P_0)

- $\Omega_+^a := \{x \in \Omega : a(x) > 0\}$. Then,

$$(A.0) \iff (P_0) \text{ has a nontrivial solution being positive in } \Omega_+^a.$$

  This is **unique** if we assume additionally

$$(A.1) \qquad \Omega_+^a = \bigcup_{j=1}^{K} \Omega_j, \quad \Omega_j \text{ is a smooth connected component},$$

  [Bandle, Pozio, Tesei, *Math. Z.*, 1988]. As a result, under $(A.0)$ and $(A.1)$, $\forall q \in \mathcal{I}_{\mathcal{N}} := \{q \in (0,1) : (P_0) \text{ has a positive solution}\}$, **a positive solution of (P_0) is unique, say $u_{\mathcal{N}} \in P^\circ$.**

- $\mathcal{A}_{\mathcal{N}} := \{q \in (0,1) : \text{ any nontrivial solution of } (P_0) \text{ is positive}\}$. Then, $\boldsymbol{\mathcal{A}_{\mathcal{N}} = (q_{\mathcal{N}}, 1)}$ for some $q_{\mathcal{N}} \in [0,1)$, [KRQU, *JDE*, 2017]. Thus actually, $\boldsymbol{(q_{\mathcal{N}}, 1) \subset \mathcal{I}_{\mathcal{N}}}$.

next, Robin case (P_α) →

Exact local multiplicity result for $\alpha > 0$ small (Robin case (P_α))

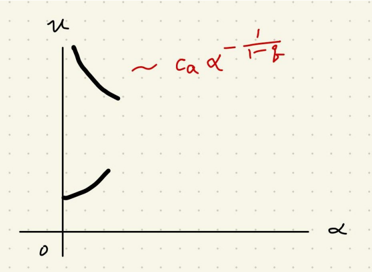
- Under $(A.0)$, [Chabrowski, Tintarev, *NoDEA*, 2014] proved the existence of **at least two nontrivial solutions** $(\alpha, u_1(\alpha))$ and $(\alpha, u_2(\alpha))$ of (P_α) for $\boldsymbol{\alpha > 0}$ **small**, which satisfy the following conditions as $\alpha \to 0^+$:

† $u_2(\alpha) \simeq c_a \alpha^{-\frac{1}{1-q}}$ as $\alpha \to 0^+$,

where $c_a = \left(\frac{-\int_\Omega a}{|\partial\Omega|} \right)^{\frac{1}{1-q}}$,

† $\exists \alpha_j \to 0^+$ s.t. $u_1(\alpha_j) \longrightarrow u_0$ in $H^1(\Omega)$,

where u_0 is a nontrivial solution of (P_0).



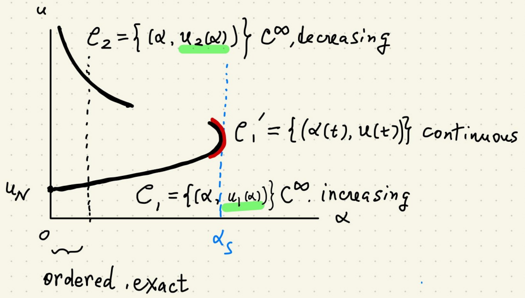
Main result 1 $\longrightarrow$

Main result 1: existence of a solution component

Theorem 1. Assume $(A.0), (A.1)$, and $q \in \mathcal{I}_{\mathcal{N}}$. Then,

$$\alpha_s := \sup\{\alpha > 0 : \ (P_\alpha) \text{ has a positive solution}\} \leq \frac{-\int_\Omega \boldsymbol{a(x)}}{\int_{\partial\Omega} \boldsymbol{u_{\mathcal{N}}^{1-q}}},$$

and (P_α) has the following two continuous curves of positive solutions.



$\alpha = \alpha_s \Longrightarrow$ a solution in P° is unique.

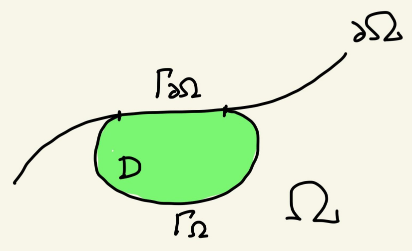
$\alpha > \alpha_s \Longrightarrow$ no solutions in P° exist.

A solution in P° is **unstable** except for $u_1(\alpha)$, $0 < \alpha \leq \alpha_s$.

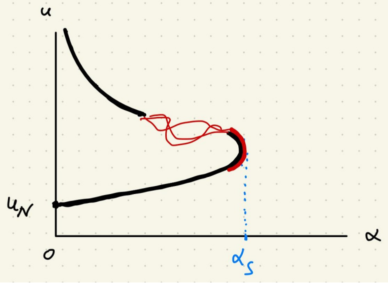
continued

Moreover, $\mathcal{C}_1'$ and $\mathcal{C}_2$ are connected by a component if we assume additionally

$$(A.2) \qquad a \geq 0 \text{ and } a \not\equiv 0 \text{ in } D \subset \Omega \text{ with } |\partial D \cap \partial\Omega| > 0.$$



(a) $(A.2)$



(b) Component of positive solutions

How to deduce the component →

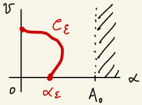
How to deduce the existence of a solution component

For $0 < \varepsilon \le \varepsilon_0$, we consider the **$\varepsilon$-regularization**

$$(P_{\alpha,\varepsilon}) \quad \begin{cases} -\Delta u = a(x) \left(\dfrac{\alpha^{\frac{1}{1-q}} u}{\alpha^{\frac{1}{1-q}} u + \varepsilon} \right)^{1-q} u^q & \text{in } \Omega, \\ \partial_\nu u = \alpha u & \text{on } \partial\Omega. \end{cases}$$

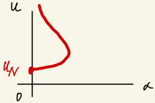
$$(P_{\alpha,\varepsilon}) \xrightarrow{\ v = \alpha^{\frac{1}{1-q}} u\ } (Q_{\alpha,\varepsilon}) \begin{cases} -\Delta v = \alpha a(x) \left(\frac{v}{v+\varepsilon} \right)^{1-q} v^q & \text{in } \Omega, \\ \partial_\nu v = \alpha v & \text{on } \partial\Omega. \end{cases}$$

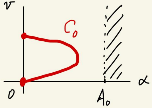
$\uparrow$ ε-regularization



$\downarrow$ $\varepsilon \to 0^+$

$$(P_\alpha) \xleftarrow{\ u = \alpha^{-\frac{1}{1-q}} v\ } (Q_\alpha) \begin{cases} -\Delta v = \alpha a(x) v^q & \text{in } \Omega, \\ \partial_\nu v = \alpha v & \text{on } \partial\Omega. \end{cases}$$





Main result 2: global exact low multiplicity result

• The Steklov eigenvalue problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega, \\ \partial_\nu \phi = \alpha\phi & \text{on } \partial\Omega \end{cases}$$

has a sequence of eigenvalues $\alpha_1 = 0 < \alpha_2 \leq \alpha_3 \leq \dots$

• With α_2 (which is *non principal*), we formulate our **global exact multiplicity** result for positive solutions of (P_α) in $0 < \alpha < \alpha_s$.

Theorem 2. Assume $(A.0), (A.1)$, and $q \in \mathcal{I}_{\mathcal{N}}$, and additionaly assume that the upper bound of α as obtained in Theorem 1 is $\leq \alpha_2$:

$$\left(\alpha_s \leq\right) \frac{-\int_\Omega a}{\int_{\partial\Omega} u_{\mathcal{N}}^{1-q}} \leq \boldsymbol{\alpha_2} \qquad \left(\xleftarrow{\text{imply}} \; q \simeq 1, \; \int_\Omega a \simeq 0 \right).$$

Then, (P_α) has **exactly two positive solutions** for each $0 < \alpha < \alpha_s$.

Remark. $0 < \alpha < \alpha_s \implies 0 < \alpha < \alpha_2$.

How to deduce Theorem 2

Scenario for the deduction of the global exactness

- Let $u \in P^\circ$ $(\neq u_1(\alpha))$ be a positive solution of (P_α) for $0 < \alpha < \alpha_s$.

$$\Longrightarrow \quad 0 < \alpha < \alpha_2, \quad \text{and} \quad u \text{ is unstable}.$$

Then, consider the following linearized eigenvalue problem at u.

$$\begin{cases} -\Delta\varphi = q\,a(x)u^{q-1}\varphi + \mu\varphi & \text{in } \Omega, \\ \partial_\nu \varphi = \alpha\varphi & \text{on } \partial\Omega, \end{cases}$$

where $\mu_1 < \mu_2 \leq \ldots,$ and we may assume $\mu_1 < 0$.

- We will verify $\boldsymbol{\mu_k \neq 0}$ **for** $\boldsymbol{\forall k \geq 2}$ (there are no zero eigenvalues).

$\Longrightarrow$ **the implicit function theorem applies** to (α, u).

$$\xrightarrow{\text{spectral analysis of } -\Delta\varphi = \lambda m(x)\varphi}$$

- Consider the general version

$$(E) \qquad \begin{cases} -\Delta\varphi = \lambda\, m(x)\varphi + \mu\varphi & \text{in } \Omega, \\ \partial_\nu \varphi = \alpha\varphi & \text{on } \partial\Omega, \end{cases}$$

where $m \in C^\theta(\overline{\Omega})$ changes sign, and $\int_\Omega m < 0$ (**having in mind** $\lambda = q$ and $m(x) = a(x)u^{q-1}$).

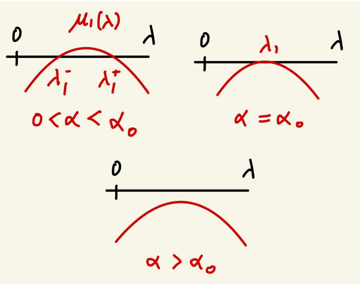
- The smallest eigenvalue $\mu_1(\lambda)$ of (E) is studied by [Afrouzi, Brown, *Proc. AMS*, 1999].

- Observe

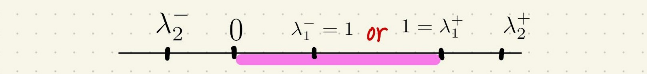
$$-\Delta u = 1 \cdot \big(a(x)\, u^{q-1}\big)\, u \text{ in } \Omega,$$

$\Longrightarrow \quad \lambda_1^- = 1, \quad \text{or} \quad \lambda_1^+ = 1.$

continued →



Proposition. If $0 < \alpha < \alpha_2$, then (E) has **no zero eigenvalue for** $\boldsymbol{0 < \lambda < \lambda_1^-}$, more precisely, $\lambda_2^- = \lambda_2^-(\alpha, m) < 0$ for any $m \in C^\theta(\overline{\Omega})$:



where $\{\lambda_k^\pm\}$ is a double sequence of eigenvalues for the eigenvalue problem

$$(E_m) \qquad \begin{cases} -\Delta\varphi = \lambda\, m(x)\varphi & \text{in } \Omega, \\ \partial_\nu \varphi = \alpha\varphi & \text{on } \partial\Omega \end{cases}$$

- Noting $\int_\Omega a u^{q-1} < 0$, the proposition shows that

$$\begin{cases} -\Delta\varphi = q\, a(x) u^{q-1}\varphi + \mu\varphi & \text{in } \Omega, \\ \partial_\nu \varphi = \alpha\varphi & \text{on } \partial\Omega \end{cases}$$

has **no zero eigenvalue**, since $0 < q < 1$ and $q \neq \lambda_1^-$.

How role $0 < \alpha < \alpha_2$ plays for the deduction of Proposition →

Sketch of proof of Proposition

Verify

$$\lambda_2^- = \lambda_2^-(\alpha, m) < 0 \quad \text{if} \quad 0 < \alpha < \alpha_2.$$

The eigenvalue problem (E_m) with $m = 1$

$$\begin{cases} -\Delta\varphi = \gamma\varphi & \text{in } \Omega, \\ \partial_\nu \varphi = \alpha\varphi & \text{on } \partial\Omega \end{cases}$$

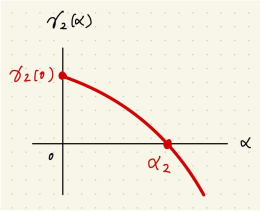
has a sequence of eigenvalues $\gamma_1 < \gamma_2 \leq \gamma_3 \leq \dots$
such that

- $\gamma_2(\alpha_2) = 0, \ \gamma_1(0) = \mathbf{0} < \boldsymbol{\gamma_2(0)}$;
- $\alpha \mapsto \gamma_2(\alpha)$ is **non increasing**.

Then,

$$0 < \alpha < \alpha_2 \quad \Longleftrightarrow \quad \gamma_2(\alpha) > 0.$$

$\xrightarrow{\text{minimax method for } \gamma_2(\alpha)}$



Let

$$\mathcal{J} = \{(A_1, A_2) : \quad A_1, A_2 \subset \Omega \text{ are disjoint and open } \},$$
$$H^1_{A_i}(\Omega) = \{\varphi \in H^1(\Omega) : \quad \varphi = 0 \quad \text{in} \quad \Omega \setminus \overline{A_i} \}.$$

Then, [Torné, *EJDE*, 2005] provides:

- $\boldsymbol{(E_1)} \quad \boldsymbol{-\Delta\varphi = \gamma\varphi}$ **in** $\boldsymbol{\Omega}$, $\quad \boldsymbol{\partial_\nu \varphi = \alpha\varphi}$ **on** $\boldsymbol{\partial\Omega}$.

$$\gamma_2(\alpha) = \min_{(A_1, A_2) \in \mathcal{J}} \max\left(\gamma^+(A_1), \quad \gamma^+(A_2)\right) \quad \text{with}$$
$$\gamma^+(A_i) = \inf\left\{\int_\Omega |\nabla\varphi|^2 - \alpha \int_{\partial\Omega} \varphi^2 : \quad \varphi \in H^1_{A_i}(\Omega), \ \|\varphi\|_2 = 1\right\}.$$

- $\boldsymbol{(E_m)} \quad \boldsymbol{-\Delta\varphi = \lambda\, m(x)\varphi}$ **in** $\boldsymbol{\Omega}$, $\quad \boldsymbol{\partial_\nu \varphi = \alpha\varphi}$ **on** $\boldsymbol{\partial\Omega}$.

  Note $\lambda_2^-(\alpha, m) = -\lambda_2^+(\alpha, -m)$, and then, similarly,

$$\lambda_2^+(\alpha, -m) = \min_{(A_1, A_2) \in \mathcal{J}} \max\left(\lambda^+(A_1), \quad \lambda^+(A_2)\right) \quad \text{with}$$
$$\lambda^+(A_i) = \inf\left\{\int_\Omega |\nabla\varphi|^2 - \alpha \int_{\partial\Omega} \varphi^2 : \quad \varphi \in H^1_{A_i}(\Omega), \ \int_\Omega m\varphi^2 = -1\right\}.$$

$\therefore$ **If** $\boldsymbol{\gamma_2(\alpha) > 0}$, **then** $\boldsymbol{\lambda_2^+(\alpha, -m) > 0}$ $\left(\because \quad \int_\Omega m\varphi^2 = -1 \implies \varphi \neq 0 \right)$.

References

- U. Kaufmann, H. Ramos Quoirin, K. U., Nonnegative solutions of an indefinite sublinear Robin problem II: local and global exactness results. *Israel J. Math.* **247**, (2022), 661–696.

- U. Kaufmann, H. Ramos Quoirin, K. U., Nonnegative solutions of an indefinite sublinear Robin problem I: positivity, exact multiplicity, and existence of a subcontinuum. *Ann. Mat. Pura Appl. (4)* **199**, (2020), 2015–2038.

Thank you for your kind attention.