5/7 '05

し

$$\boxed{\cdot\ \frac{dy}{dx} = e^{x-y}}$$

$$\int e^y\,dy = \int e^x\,dx$$

$$\therefore\ e^y = e^x + C \qquad \therefore\ \underline{y = \log(e^x + C)}\,.$$

$$\boxed{\cdot\ \frac{dy}{dx} = \frac{xy}{x^2-1}}$$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{(x^2-1)'}{x^2-1}\,dx$$

$$= \frac{1}{2}\log|x^2-1| + C$$

$$\therefore\ \log|y| = \frac{1}{2}\log|x^2-1| + C$$

$$\therefore\ 2\log|y| = \log|x^2-1| + C$$

$$\log y^2 = \log|x^2-1| + C$$

$$\log\left|\frac{y^2}{x^2-1}\right| = C \qquad \therefore\ \frac{y^2}{x^2-1} = C$$

$$\therefore\ \underline{y^2 = C(x^2-1)}$$

$$\boxed{\cdot\ \frac{dy}{dx} = y(1-y)}$$

$$\int \frac{1}{y(1-y)}\,dy = \int dx$$

$$\int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int dx$$

$$\therefore\ \log|y| - \log|1-y| = x + C$$

$$\therefore\ \log\left|\frac{y}{1-y}\right| = x + C$$

$$\therefore\ \frac{y}{1-y} = Ce^x$$

yについて解くと $\underline{y = \dfrac{Ce^x}{1+Ce^x}}$.

$$\boxed{\cdot\ \frac{dy}{dx} = e^{y}\sin x}$$

$$\int e^{-y}\,dy = \int \sin x\,dx$$

$$-e^{-y} = -\cos x + C \qquad \therefore\ y = -\log(\cos x + C)$$

同次形

$$\boxed{\cdot\ \frac{dy}{dx} = \frac{x+y}{x-y}} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}.$$

$y/x = z$ とおくと,

$$\frac{dz}{dx} = \frac{1}{x}\cdot\frac{1+z^2}{1-z}$$

$$\int \frac{1-z}{1+z^2}\,dz = \int \frac{1}{x}\,dx$$

$$\therefore\ \int \frac{1}{1+z^2}\,dz - \int \frac{z}{1+z^2}\,dz = \int \frac{1}{x}\,dx.$$

$$\tan^{-1} z - \frac{1}{2}\log(1+z^2) = \log|x| + C$$

$$\therefore\ 2\tan^{-1}\left(\frac{y}{x}\right) = \log(x^2+y^2) + C$$

$$\boxed{\cdot\ \frac{dy}{dx} = \frac{x^2+y^2}{2xy}} = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad z = \frac{y}{x}$$

$\therefore\ \dfrac{dy}{dx} = z + x\dfrac{dz}{dx}$ より

$$z + x\frac{dz}{dx} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\therefore\ \frac{dz}{dx} = \frac{1}{x}\cdot\frac{1-z^2}{2z}$$

$$\therefore\ \int \frac{2z}{1-z^2}\,dz = \int \frac{1}{x}\,dx$$

$\therefore \int \left(\frac{1}{1-z} - \frac{1}{1+z} \right) dz = \int \frac{1}{x} dx$

$\therefore -\log|1-z| - \log|1+z| = \log|x| + C$

$\therefore \quad C = \log|(1-z)(1+z)\cdot x|$

$\therefore \quad (1-z)(1+z)\cdot x = C$

$\therefore \quad x - \frac{y^2}{x} = C.$

$\boxed{\cdot \quad \frac{dy}{dx} = \sqrt{x+y+1} \quad (z = x+y+1).}$

$z = x+y+1$ より

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$ だから $-1 + \frac{dz}{dx} = \sqrt{z}$

$\therefore \quad \frac{dz}{dx} = 1 + \sqrt{z} \qquad \therefore$

$\therefore \quad \int \frac{1}{1+\sqrt{z}} dz = \int dx$

$\sqrt{z} = t$ とおいて置換積分（微積テキスト, p.78 (1)）

すると $\int \frac{1}{1+\sqrt{z}} dz = \int \frac{2t}{1+t} dt = \int \left(2 - \frac{2}{1+t} \right) dt$

$= 2t - 2\log|1+t| + C$

$= 2\sqrt{z} - 2\log(1+\sqrt{z}) + C$

$\therefore \quad 2\sqrt{z} - 2\log(1+\sqrt{z}) = x + C.$

$\therefore \quad 2\sqrt{x+y+1} - 2\log(1+\sqrt{x+y+1}) = x + C.$

$\boxed{\dfrac{dy}{dx} = y + e^{-x}}$ $\quad \dfrac{dy}{dx} = y$ より $y = Ce^{x}$.

$y = c(x)e^{x}$ とおいて, 代入. $c'(x)e^{x} + c(x)e^{x} = c(x)e^{x} + e^{-x}$

$\therefore\ c'(x) = e^{-2x} \quad \therefore\ c(x) = -\frac{1}{2}e^{-2x} + C$.

$\therefore\ y = \left(-\frac{1}{2}e^{-2x} + c\right)e^{x} = \underline{-\frac{1}{2}e^{-x} + ce^{x}}$.

$\boxed{\dfrac{dy}{dx} + 2xy = x}$ $\quad \dfrac{dy}{dx} = -2xy$ より $y = Ce^{-x^2}$

$y = C(x)e^{-x^2}$ とおくと, $y' = c'(x)e^{-x^2} + c(x)(-2x)e^{-x^2}$

代入すると, $c'(x)e^{-x^2} = x \quad \therefore\ c'(x) = xe^{x^2}$.

置換積分より $c(x) = \frac{1}{2}e^{x^2} + C \quad \therefore\ y = \left(\frac{e^{x^2}}{2} + c\right)e^{-x^2}$

$= \underline{\frac{1}{2} + ce^{-x^2}}$.

$\boxed{x\dfrac{dy}{dx} + y = \cos x}$ $\quad x\dfrac{dy}{dx} + y = 0$ より

$\int \frac{1}{y}dy = -\int \frac{1}{x}dx \quad \therefore\ \log|y| + \log|x| = C$

$\therefore\ \log|xy| = C \quad \therefore\ xy = C \quad \therefore y = \dfrac{C}{x}$.

$y = \dfrac{C(x)}{x}$ とおくと, $\dfrac{dy}{dx} = \dfrac{c'(x)x - c(x)}{x^2}$. 代入,

$x \cdot \dfrac{c'(x)x - C(x)}{x^2} + \dfrac{C(x)}{x} = \cos x \quad \therefore\ c'(x) = \cos x$.

$\therefore\ c(x) = \sin x + C \quad \therefore\ y = \dfrac{\sin x + C}{x} = \underline{\dfrac{\sin x}{x} + \dfrac{C}{x}}$.

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$\boxed{\cos x \dfrac{dy}{dx} - y\sin x = x}$ $\cos x \dfrac{dy}{dx} - y \sin x = 0$ より

$\dfrac{dy}{dx} = y\tan x$. ∴ $\int \dfrac{1}{y}\,dy = \int \tan x\,dx$.

$\int \tan x\,dx = -\int \dfrac{(\cos x)'}{\cos x}\,dx = -\log|\cos x| + C$

∴ $\log|y| + \log|\cos x| = C$ ∴ $\log|y\cos x| = C$ ∴ $y\cos x = C$

$y = \dfrac{C(x)}{\cos x}$ とおく. $y' = \dfrac{C'(x)\cos x + C(x)\sin x}{\cos^2 x}$. 代入.

$C'(x) = x$ ∴ $C(x) = \dfrac{x^2}{2} + C$. ゆえに,

$y = \dfrac{\frac{x^2}{2} + C}{\cos x} = \underline{\dfrac{x^2}{2\cos x} + \dfrac{C}{\cos x}}$.

$\boxed{\dfrac{dy}{dx} + \dfrac{y}{x} = \log x}$ $\dfrac{dy}{dx} + \dfrac{y}{x} = 0$ より $y = \dfrac{C}{x}$.

$y = \dfrac{C(x)}{x}$ とおく. $y' = \dfrac{C'(x)x - C(x)}{x^2}$. 代入,

$\dfrac{C'(x)}{x} = \log x$ ∴ $C'(x) = x\log x$. 部分積分により

$C(x) = \dfrac{x^2}{2}\log x - \dfrac{x^2}{4} + C$ ∴ $y = \underline{\dfrac{x}{2}\log x - \dfrac{x}{4} + \dfrac{C}{x}}$

$\boxed{\dfrac{dy}{dx} + \dfrac{y}{x} = x}$ $\dfrac{dy}{dx} + \dfrac{y}{x} = 0$ より $y = \dfrac{C}{x}$.

$y = \dfrac{C(x)}{x}$ より $y' = \dfrac{C'(x)x - C(x)}{x^2}$. 代入,

$C'(x) = x^2$ ∴ $C(x) = \dfrac{x^3}{3} + C$ ∴ $y = \underline{\dfrac{x^2}{3} + \dfrac{C}{x}}$

微分方程式 演習問題

6/4 '05

1. 次の方程式の一般解を求めよ.（方程式のタイプをまず考えよ）

(1) $$\frac{dy}{dx} = y^2 - 4xy^2$$

(2) $$\frac{dy}{dx} = \frac{x}{x+y}$$

(3) $$\frac{dy}{dx} = -\frac{y}{x} + \sin x$$

(4) $$\frac{dy}{dx} = -y + xy^3$$

2. 次の方程式を考える

$$\frac{dy}{dx} = y^2 - \frac{2}{x^2}$$

(1) $y = \frac{1}{x}$ は解であることを示せ.

(2) $z = y - \frac{1}{x}$ で変換するならば

z についてベルヌーイ型になることを導け.

(3) $y(1) = 2$ の初期条件のもとで

特殊解を求めよ.

6/4 解答

1 (1) 変数分離形 $y = \dfrac{1}{2x^2 - x + C}$

(2) 同次形

$$(1+\sqrt{5})\log\left|\frac{y}{x} + \frac{1-\sqrt{5}}{2}\right| - (1-\sqrt{5})\log\left|\frac{y}{x} + \frac{1+\sqrt{5}}{2}\right|$$

$$= -2\sqrt{5}\log|x| + C$$

(3) 1階線形 $y = \dfrac{-x\cos x + \sin x + C}{x}$

(4) ベルヌーイ型 $y^2 = \dfrac{2}{2x + 1 + Ce^{2x}}$

2. (1) 直接代入せよ.

(2) $\dfrac{dz}{dx} = \dfrac{2}{x} z + z^2$ ($n=2$ のベルヌーイ型)

(3) $y = \dfrac{1}{x} + \dfrac{3x^2}{-x^3 + 4}$.