問　$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$ を対角化せよ.

答　$|\lambda E_2 - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda + 2 \end{vmatrix} = (\lambda - 1)(\lambda + 2) - 4$

$= \lambda^2 + \lambda - 6$

$= (\lambda + 3)(\lambda - 2) = 0.$

$\therefore \lambda = -3, 2$ (固有値)

それぞれの固有空間は,

$((-3)E_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

拡大, $\left(\begin{array}{cc:c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right) \to \to \cdots \to \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc:c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-1/2) \cdot t \\ t \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}.$

よって固有空間は

$V(-3) = \underline{V\left(\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}\right)}$

$(2E_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

拡大, $\left(\begin{array}{cc:c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right) \to \cdots \to \left(\begin{array}{cc:c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

$\therefore$ 固有空間 $V(2) = \underline{V\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)}.$

ここで

$$P = \begin{pmatrix} -1/2 & 2 \\ 1 & 1 \end{pmatrix}$$

とおくと，P は正則で，

$$P^{-1} A P = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$$

である．