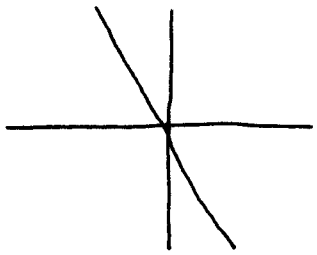
問　$l : y = -4x$ 上の $x = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$ の正射影を求めよ.

解



$y = -4x$

$l : V\left(\begin{pmatrix} 1 \\ -4 \end{pmatrix}\right)$

$\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ を正規化 $\Rightarrow \dfrac{1}{\sqrt{17}}\begin{pmatrix} +1 \\ -4 \end{pmatrix}$.

$\therefore\ l : V\left(\underbrace{\begin{pmatrix} 1/\sqrt{17} \\ -4/\sqrt{17} \end{pmatrix}}_{\text{ONB}\ \because\ m_1}\right)$

よって. x の正射影 x_p は

$$x_p = (x, m_1)m_1 = \left(\begin{pmatrix} -2 \\ -6 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{17} \\ -4/\sqrt{17} \end{pmatrix}\right) \cdot \begin{pmatrix} 1/\sqrt{17} \\ -4/\sqrt{17} \end{pmatrix}$$

$$= \frac{1}{17}(-2+24)\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 22/17 \\ -88/17 \end{pmatrix}.$$

問　$V = V\left(\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right)$ 上の $x = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ の正射影を求めよ.

解　$a_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ として, $m_1 = \dfrac{1}{|a_1|}a_1 = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

a_2 の m_1 方向の正射影は

$$(a_2)_p = (a_2, m_1)m_1 = \frac{1}{5}(-1)\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 0 \\ -2/5 \end{pmatrix}.$$

$\therefore$ 余ベクトル $(a_2)_\perp$ は

$$(a_2)_\perp = a_2 - (a_2)_p = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1/5 \\ 0 \\ -2/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 6 \\ 5 \\ \cancel{-7}\ -3 \end{pmatrix}.$$

$\begin{pmatrix} 6 \\ 5 \\ \cancel{-7}\ -3 \end{pmatrix}$ を正規化して $m_2 = \cancel{\dfrac{1}{\sqrt{36+25+49}}\begin{pmatrix} 6 \\ 5 \\ -7 \end{pmatrix}}$. $\underbrace{\dfrac{1}{\sqrt{36+25+9}}\begin{pmatrix} 6 \\ 5 \\ -3 \end{pmatrix}}_{1/\sqrt{70}\begin{pmatrix} 6 \\ 5 \\ -3 \end{pmatrix}}$.

$$\therefore\ V = V\left(\underbrace{\begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 6/\sqrt{70} \\ 5/\sqrt{70} \\ -3/\sqrt{70} \end{pmatrix}}_{\text{ONB}}\right)$$

よって,

$$x_p = (x, m_1)m_1 + (x, m_2)m_2$$

$$= \left(\begin{pmatrix}4\\-2\\3\end{pmatrix}, \begin{pmatrix}1/\sqrt{5}\\0\\2/\sqrt{5}\end{pmatrix}\right)\begin{pmatrix}1/\sqrt{5}\\0\\2/\sqrt{5}\end{pmatrix} + \left(\begin{pmatrix}4\\-2\\3\end{pmatrix}, \begin{pmatrix}6/\sqrt{70}\\5/\sqrt{70}\\-3/\sqrt{70}\end{pmatrix}\right)\begin{pmatrix}6/\sqrt{70}\\5/\sqrt{70}\\-3/\sqrt{70}\end{pmatrix}$$

$$= \frac{1}{5}(4+6)\begin{pmatrix}1\\0\\2\end{pmatrix} + \frac{1}{70}(24-10-9)\begin{pmatrix}6\\5\\-3\end{pmatrix}$$

$$= \begin{pmatrix}2\\0\\4\end{pmatrix} + \frac{1}{14}\begin{pmatrix}6\\5\\-3\end{pmatrix} = \underline{\frac{1}{14}\begin{pmatrix}34\\5\\53\end{pmatrix}}.$$

問　$V: x-y+2z=0$ 上の $x=\begin{pmatrix}-1\\5\\2\end{pmatrix}$ の正射影を求めよ.

答　パラメータ t, s を $y=t,\ z=s$ とおくとき, $x=t-2s$.

$\therefore$ V 上の点, $\begin{pmatrix}x\\y\\z\end{pmatrix}$ は,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}t-2s\\t\\s\end{pmatrix} = t\begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}-2\\0\\1\end{pmatrix} \qquad \therefore V = V\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right).$$

V の ONB を求めよう. $a_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $a_2 = \begin{pmatrix}-2\\0\\1\end{pmatrix}$ とおいて,

$$m_1 = \frac{1}{|a_1|}a_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

a_2 の m_1 方向の正射影を求めると,

$$(a_2)_p = (a_2, m_1)m_1 = \left(\begin{pmatrix}-2\\0\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}(-2)\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\-1\\0\end{pmatrix}.$$

余ベクトル $(a_2)_\perp$ は

$$(a_2)_\perp = a_2 - (a_2)_p = \begin{pmatrix}-2\\0\\1\end{pmatrix} - \begin{pmatrix}-1\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

これを正規化して,

$$m_2 = \frac{1}{|a_2|}a_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\therefore V = V\left(\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}, \begin{pmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\right)$$

よって $\mathbf{x}$ の V 上の正射影 $\mathbf{x}_p$ は

$$\mathbf{x}_p = (\mathbf{x}, \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}, \mathbf{u}_2)\mathbf{u}_2$$

$$= \left(\begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left(\begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

$$= \frac{1}{2}\underbrace{(-1+5)}_{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3}\underbrace{(1+5+2)}_{8/3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -2 \\ 14 \\ 8 \end{pmatrix}.$$