問 $$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

について $(x,y)=(0,0)$ において $f_{xy}(0,0) \neq f_{yx}(0,0)$ を導け.

答 $$f_{xy}(0,0) = (f_x)_y(0,0)$$

$$= \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}.$$

$(x,y) \neq (0,0)$ のとき,

$$f_x(x,y) = \frac{\{y(x^2-y^2) + xy\cdot 2x\}(x^2+y^2) - xy(x^2-y^2)\cdot 2x}{(x^2+y^2)^2}.$$

$$\therefore f_x(0,h) = \frac{-h^3\cdot h^2 - 0}{h^4} = -h$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{-h}{h} = -1.$$

一方, $$f_{yx}(0,0) = (f_y)_x(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$(x,y) \neq (0,0)$ のとき

$$f_y(x,y) = \frac{\{x(x^2-y^2) + xy(-2y)\}(x^2+y^2) - xy(x^2-y^2)\cdot 2y}{(x^2+y^2)^2}.$$

$$\therefore f_y(h,0) = \frac{h^5 - 0}{h^4} = h$$

$$\therefore f_{yx}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$\therefore f_{xy}(0,0) \neq f_{yx}(0,0).$$

例　$f(x,y) = \dfrac{x^2y}{x^2+y^2}$ の $(2,\ -1,\ f(2,-1))$ における接平面の方程式を求めよ.

答　$$f_x = \frac{2xy(x^2+y^2) - x^2y\cdot 2x}{(x^2+y^2)^2} \qquad \therefore\ f_x(2,-1) = -\frac{4}{25}$$

$$f_y = \frac{x^2(x^2+y^2) - x^2y\cdot 2y}{(x^2+y^2)^2} \qquad \therefore\ f_y(2,-1) = \frac{12}{25}$$

求める方程式は

$$g(x,y) = -\frac{4}{25}(x-2) + \frac{12}{25}(y+1) + \underbrace{f(2,-1)}_{-4/5}.$$

$$= -\frac{4}{25}x + \frac{12}{25}y.$$