[問] $\displaystyle\int_0^1 \left(\int_{x-1}^{-x+1} xy\, dy \right) dx$ の 積分順序を 変換して
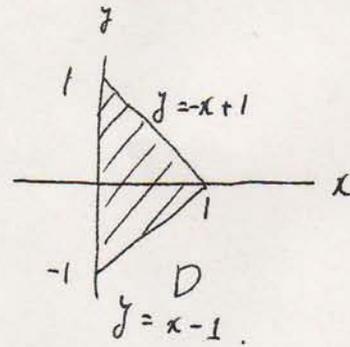
計算せよ。

[答]

$$= \int_{-1}^0 \left(\int_0^{y+1} xy\, dx \right) dy$$

$$+ \int_0^1 \left(\int_0^{-y+1} xy\, dx \right) dy$$

$$= \int_{-1}^0 \left[\frac{x^2 y}{2} \right]_0^{y+1} dy + \int_0^1 \left[\frac{x^2 y}{2} \right]_0^{-y+1} dy$$

$$= \int_{-1}^0 \frac{(y+1)^2 y}{2} dy + \int_0^1 \frac{(-y+1)^2 y}{2} dy$$

$$= \left[\underbrace{\frac{(y+1)^3}{6} y}_{0} \right]_{-1}^0 - \frac{1}{6}\int_{-1}^0 (y+1)^3 dy \ + \left[\underbrace{\frac{(-y+1)^3}{6} y}_{0} \right]_0^1 + \frac{1}{6}\int_0^1 (-y+1)^3 dy$$

$$= -\frac{1}{24}\left[(y+1)^4 \right]_{-1}^0 + \frac{1}{24}\left[(-y+1)^4 \right]_0^1$$

$$= -\frac{1}{24} + \frac{1}{24}(0-1) = 0$$